# CHAPTER 7 BOOLEAN SEMANTICS FOR PLURALITY

## 7.1. Boolean algebras

Boolean algebras are structures that model what could be called
**naturalistic typed extensional part-of** structures.

Think of me and my parts. Different things can be part of me in different senses.
For instance, I can see myself as the whole of my body parts, but also as the whole of my achievements, and in many other ways.

-**Typing** restricts the notion of part-of to one of those senses. Thus, the domain in which I am the whole of my body parts will not include my achievements as part of me, even though, in another sense these **are** part me.

-**Extensionality** means that **in the domain chosen** I am not more and not less than the whole of my parts.
This means that if I choose the domain in which I am the whole of my body parts, then I cannot express **in that domain** that really and truly I am more than the sum of my parts.
If I want to express the latter, I must do that by **relating** the whole of my body parts in the body part domain to me **as an entity in a different domain** in which I cannot be divided into body parts.
This is how **intensionality** comes in: I can be at the same time indivisible me **and** the sum of my body parts, because I occur in different domains (or, alternatively, there is a cross-domain identity relation which identifies objects in different domains as me).

-**Naturalistic** means that each typed extensional part-of notion is naturalistic: in that structure parts work the way parts naturalistically work.
The claim is, then, that Boolean algebras formalize this notion.

I define a successive series of notions:

A **partial order** is a pair **B** = $\langle B, \sqsubseteq \rangle$, where B is a non-empty set and $\sqsubseteq$, the **part-of** relation, is a reflexive, transitive, antisymmetric relation on B.

Reflexive: for every $b \in B$: $b \sqsubseteq b$
Transitive: for every $a,b,c \in B$: if $a \sqsubseteq b$ and $b \sqsubseteq c$ then $a \sqsubseteq c$
Antisymmetric: for every $a,b \in B$: if $a \sqsubseteq b$ and $b \sqsubseteq a$ then $a = b$

In a partial order we define notions of **join, $\sqcup$,** and **meet**, $\sqcap$. In a structure of parts, we think of join as the **sum** of the parts, and of meet as their **overlap**.

Let **B** = $\langle B, \sqsubseteq \rangle$ be a partial order and $a,b \in B$:

The **join** of a and b, $(a \sqcup b)$, is the unique element of B such that:
1. $a \sqsubseteq (a \sqcup b)$ and $b \sqsubseteq (a \sqcup b)$.
2. For every $c \in B$: if $a \sqsubseteq c$ and $b \sqsubseteq c$ then $(a \sqcup b) \sqsubseteq c$.

The **meet** of a and b, $(a \sqcap b)$, is the unique element of B such that:
1. $(a \sqcap b) \sqsubseteq a$ and $(a \sqcap b) \sqsubseteq b$.
2. For every $c \in B$: if $c \sqsubseteq a$ and $c \sqsubseteq b$ then $c \sqsubseteq (a \sqcap b)$

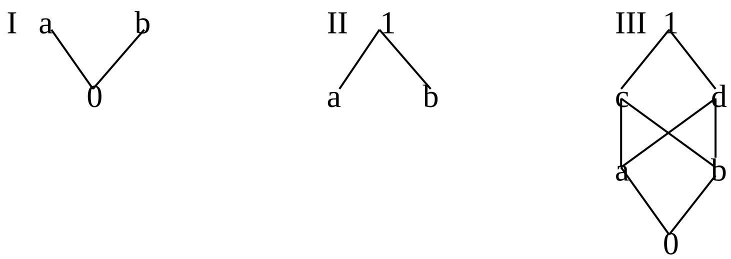
The idea is:
-To find the join of a and b, you look **up** in the part-of structure at the set of all elements of which both a and b are part: $\{c \in B: a \sqsubseteq c \text{ and } b \sqsubseteq c\}$.
The join of a and b is the **minimal** element in that set.

-To find the meet of a and b, you look **down** in the part-of structure at the set of all elements which are part both of a and of b: $\{c \in B: c \sqsubseteq a \text{ and } c \sqsubseteq b\}$.
The meet of a and b is the **maximal** element in that set.

Elements need not have a join or a meet in a partial order. A partial order in which any two elements **do** have a both a join and a meet is called a **lattice**:

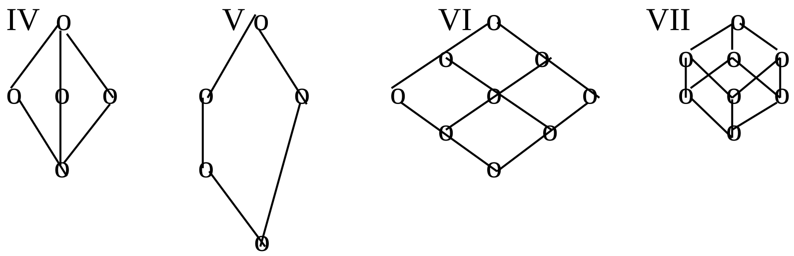
A **lattice** is a partial order **B** = $\langle B,\sqsubseteq\rangle$ such that:
for every $a,b \in B$: $(a \sqcup b), (a \sqcap b) \in B$.

Not lattices:




In I a and b do not have a join.
In II a and b do not have a meet.
In III, the set $\{c \in B: a \sqsubseteq c \text{ and } b \sqsubseteq c\} = \{c,d,1\}$.
This set doesn't have a minimum, hence a and b don't have a join.
Similarly, c and d don't have a meet.

Lattices:




A partial order **B** = $\langle B,\sqsubseteq\rangle$ is **bounded** iff **B** has a **maximum** 1 and a **minimum** 0 in **B**.
1 is a **maximum** in **B** iff $1 \in B$ and for every $b \in B$: $b \sqsubseteq 1$
0 is a **minimum** in **B** iff $0 \in B$ and for every $b \in B$: $0 \sqsubseteq b$

Obviously, then, a bounded lattice is a lattice which is bounded.

**Fact**: Every finite lattice is bounded.

**Proof Sketch :** If B = $\{b_1,...,b_n\}$ then $1 = (b_1 \sqcup (b_2 \sqcup (... \sqcup (b_{n-1} \sqcup b_n)...)$ and
$0 = (b_1 \sqcap (b_2 \sqcap (... \sqcap (b_{n-1} \sqcap b_n)...)$

Not every lattice is bounded, though.
If we take Z, the integers, ...,–1,0,1,... with the normal partial order $\leq$ and we define:

$(n \sqcap m)$ = the smallest one of n and m
$(n \sqcup m)$ = the biggest one of n and m

then we get a lattice, but not one that has a minimum or a maximum.

0 is part of everything.
Conceptually this means that 0 doesn't count as a **naturalistic** part.
Nevertheless, it is **very** convenient to admit 0 to the structure, because it simplifies proofs of structural insights immensely.
But indeed we do not count 0 as a naturalistic part, because we **define**:

a and b are **disjoint**, have **no part in common** iff $(a \sqcap b)=0$
a and b **overlap** iff $(a \sqcap b)\neq 0$

If $X \subseteq B$, where B is a lattice, then $X^+ = X-\{0\}$.

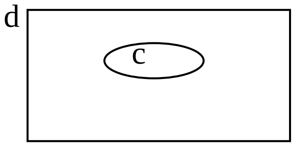
I will here call the elements of $B^+$ the **objects** in B.
Thus, if objects a and b overlap they have a part that is itself an object.

We assume as minimal constraints on naturalistic extensional typed part-of structures that they be lattices with a minimum 0.
And this means that we assume that in a part-of structure we can always carve out the overlap of two parts;
If a and b overlap, the object $a \sqcap b$ is their overlap;
If a and b are disjoint, their overlap is element 0.

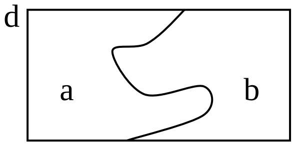
Now we come to the naturalistic constraints.
They concern what happens to the parts of an object a when we cut it.
When I say cut it, I mean **cut** it.
In a naturalistic part-of structure, cutting should be **real cutting**, the way my daughter used to do it.
It turns out that real cutting imposes requirements on part-of structures that not all part of structures allowed so far can live up to. If in those structures cutting cannot mean real cutting, they cannot be regarded as naturalistic structures.

**Naturalistic constraint on cutting 1**:

Suppose we have object d and part c of d:




Now suppose we cut through d:



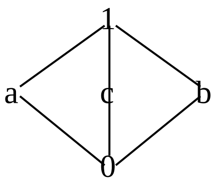

This means that $d = (a \sqcup b)$ and $(a \sqcap b) = 0$.
If we cut in this way, where is c going to sit?
And the intuitive answer is obvious:

> **either** c is part of a, and c goes to the a-side,
> **or** c is part of b, and c goes to the b-side,
> **or** c overlaps both a and b and hence **c itself** gets cut into a part $a_1$ of a and a part $b_1$ or b, and this means that $c = a_1 \sqcup b_1$ and $a_1 \sqcap b_1 = 0$.

Structures that don't satisfy this naturalistic cutting intuition miss something crucial about what parts are and how you cut. We define:

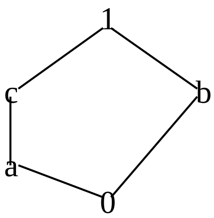
Lattice **B** $= \langle B,\sqsubseteq\rangle$ is **distributive** iff for every $a,b,c \in B$:
If $c \sqsubseteq (a \sqcup b)$ then either $c \sqsubseteq a$ or $c \sqsubseteq b$
or for some $a_1 \sqsubseteq a$ and some $b_1 \sqsubseteq b$: $c = (a_1 \sqcup b_1)$

Lattices IV and V are not distributive. IV is called the **diamond**, V is called the **pentagon**.
The diamond is not distributive:




$1 = (a \sqcup b)$, but c is not part of a, nor of b, nor do a and b have parts $a_1$ and $b_1$ respectively, such that c is the sum of $a_1$ and $b_1$.

The pentagon is not distributive:




$1 = (a \sqcup b)$ but c is not part of a, not of b, and not the sum of any part of a and any part of b.

There is a very insightful theorem about distributivity, which says that in essence, the pentagon and the diamond are the only possible sources of **non-distributivity**:

**Theorem:** A lattice is distributive **iff** you cannot find either the pentagon or the diamond as a substructure.

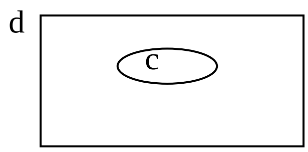
Standardly, distributivity is defined by one of the following two alternative definitions:

(1) Lattice $\mathbf{B} = <B,\sqsubseteq>$ is distributive iff
for all $a,b,c \in B$: $(a \sqcap (b \sqcup c)) = ((a \sqcap b) \sqcup (a \sqcap c))$
(2) Lattice $\mathbf{B} = <B,\sqsubseteq>$ is distributive iff
for all $a,b,c \in B$: $(a \sqcup (b \sqcap c)) = ((a \sqcup b) \sqcap (a \sqcup c))$

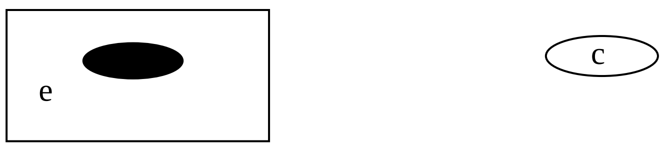
I gave the above definition because its conceptual naturalness is clearer.
But it is not hard to prove that either definition (1) or (2) is equivalent to the one I gave.

We come to the second constraint on cutting:

**Naturalistic constraint on parts and cutting 2**:
We go back to d and c:




The second naturalistic intuition is very simple.
It is that c can indeed be cut out of d, and that cutting it out means what it intuitively does: when you cut c out of d, you remove from the partial order all parts of d that have a part in common with c.
This means that you will be left only with parts of d that didn't have a part in common with c.
The naturalistic constraint is that what you are left with is a part e of d such that
$c \sqcup e = d$ and $c \sqcap e = 0$, and **its** parts.
What this says is that the way my daughter cuts is like this:



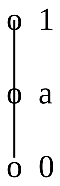

The constraint, thus, **regulates** in cutting the relation between what you cut out and what is left. What is left, when we cut out c, we call the complement of c.
We define:

Bounded lattice $\mathbf{B} = <B,\sqsubseteq>$ is **complemented** iff
for every $b \in B$: there is a $c \in B$ such that $(b \sqcup c) = 1$ and $(b \sqcap c) = 0$.

In **non-**distributive bounded lattices elements can have more than one complement.
For instance, in the diamond, both a and b are complement of c.

In **distributive** bounded lattices each element has at most one complement.  Thus each element has null or one complement.
This means that in a complemented distributive lattice each element has a unique complement:

Example of a distributive lattice that is not complemented:




**a** is a non-null proper part of 1 (a part that is smaller than1b itself).  **a** does not have a complement.
The notion of complement naturally generalizes from 1 to arbitrary non-null elements in B:

Let a,b ∈ B − {0}, and a ⊑ b

The **relative complement of** a **in** b, $\neg_b(a)$ is the unique element of B such that  a ⊓ $\neg_b(a)$ = 0 and a ⊔ $\neg_b(a)$ = b  (if there is one)

The relative complement of a in b represents the natural notion of 'remainder'.
The naturalistic intuition is that if a is part of b, but not all of b, then there is something left, and the maximal part of b that doesn't overlap a is a's relative complement.

All this motivates the assumption of compementation for naturalistic structures:

**B** = <B,⊑> is a **Boolean lattice** iff  **B** is a complemented distributive lattice.

Next we generalize the notions of meet and join:

Let **B** = <B,⊑> be a partial order and X ⊆ B.

The **join** of X in **B** is the unique element ⊔X of B such that:
1. for every x ∈ X:  x ⊑ ⊔X
2. for every c ∈ B: if for every x ∈ X: x ⊑ c then ⊔X ⊑ c

The **meet** of X in **B** is the unique element ⊓X of B such that:
1. for every x ∈ X: ⊓X ⊑ x
2. for every c ∈ B: if for every x ∈ X: c ⊑ x then c ⊑ ⊓X

Lattice **B** = <B,⊑> is **complete** iff for every X ⊆ B: ⊔X, ⊓X ∈ B.

In a complete lattice not just finite joins and meets exist, but all joins and meets do.

A **powerset lattice** is a structure **pow(A)**  = <**pow**(A), ∪, ∩>, for some set A.

**Facts**:  Every finite lattice is complete.
Every powerset lattice is complete.

We have already indicated why the first fact holds above:
for finite lattices $\sqcup X$ and $\sqcap X$ can be defined in terms of the two place join and meet.

A powerset lattice is complete because its domain is the set of **all** subsets of some set A, and for any set $X \subseteq$ **pow**(A), $\cup X$ and $\cap X$ are subsets of A, hence in **pow**(A).

Not every lattice is complete, not every Boolean lattice is complete.

**Example:**
Let A be an infinite set.
FIN(A) is the set of all finite subsets of A.
COFIN(A) is the set of all subsets whose settheoretic complement in A is finite:

$$\text{COFIN}(A) = \{X \subseteq A: A{-}X \text{ is finite}\}$$

(So if A = N, the set of natural numbers, N−{1,2,3} is cofinite, but E, the set of even numbers is not.).

Let B = FIN(A) $\cup$ COFIN(A), and look at: $\langle B, \subseteq \rangle$.
This is a bounded distributive lattice with minimum Ø and maximum A, and it is complemented as well: For every $X \subseteq A$: $\neg(X) = A - X$.
Namely: if $X \in B$ then $X \subseteq A$ and X is finite or X is cofinite

if X is finite, , then A – X is cofinite, hence $A - X \in B$. If X is cofinite,then A – X is finite, hence $A - X \in B$. This means that B is a Boolean lattice.

But B is not complete. For example:

Every singleton set is finite, so $\{\{n\}: n \text{ is even}\} \subseteq$ FIN(A), hence $\{\{n\}: n \text{ is even}\} \subseteq B$.
But $\cup\{\{n\}: n \text{ is even}\}$ = E, the set of even numbers, and E is not in B, it is neither finite nor cofinite.

For practically all purposes of semantics we assume our structures to be complete.
For complements, completeness has the intuitive consequence that:

**Fact**: Every complete Boolean lattice is relatively complemented, which means that:
for every $a,b \in B^+$: if $a \sqsubseteq b$ then $\neg_b(a) \in B$

(Proof: $\neg_b(a) = \sqcup\{c \sqsubseteq b: a \sqcap c = 0\}$.
Since, $\{c \sqsubseteq b: a \sqcap c = 0\} \subseteq B$, this join is in B, if B is complete.)

Notice that if we assume our structure to be complete distributive lattices, we can do with a very weak complement condition to get Boolean lattices.

We define the part set of an element b of B:

$$(b] = \{x \in B: x \sqsubseteq b\}.$$

**Complementation in complete distributive lattices:**
If $a \in (b]^+$ then there is a $c \in (b]^+ : a \sqcap c = 0$

If you take out something (real) from b (namely a), there is something (real_ left.

One more central notion is that of atomicity:

Let **B** = <B,$\sqsubseteq$> be a partial order with minimum 0 and let a $\in$ B.
a is an **atom** in **B** iff a $\in$ $B^+$ and for every b $\in$ $B^+$: if b $\sqsubseteq$ a then either b = a

a is an atom in **B** if there is nothing in between 0 and a.

Let **B** = <B, $\sqsubseteq$> be a partial order with minimum 0.
**B** is **atomic** iff for every b $\in$ $B^+$: there is an atom a $\in$ B such that a $\sqsubseteq$ b.

Thus, in an atomic partial order, every element has an atomic part.
*De facto* this means that when you divide something into smaller and smaller parts, you **always** end up with atoms.

**Facts:** Every finite lattice is atomic.
Every powerset lattice is atomic.

That every finite lattice is atomic is obvious when you think about it: in order for an element not to have any atomic parts, you need to find smaller and smaller parts all the way down.
That implies infinity.
In the powerset lattice <**pow**(A),$\subseteq$> the singleton sets are the atoms.

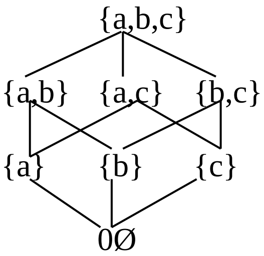
Not every infinite lattice is atomic.
In fact, infinite lattices, and for that matter, infinite **Boolean** lattices exist that are **atomless**, that have no atoms at all.
Even stronger, **complete** Boolean lattices exist that are atomless.

We have defined Boolean lattices. We now define **Boolean algebras**:

A **Boolean algebra** is a structure **B** = <B, $\sqsubseteq$, $\neg$, $\sqcap$, $\sqcup$, 0, 1> such that:
1. <B,$\sqsubseteq$> is a **Boolean lattice**.
2. $\neg$: B $\rightarrow$ B is the operation which maps every b $\in$ B onto its complement $\neg$b
3. $\sqcap$: (B$\times$B)$\rightarrow$B is the operation which maps every a,b $\in$ B onto their meet (a $\sqcap$ b)
4. $\sqcup$: (B$\times$B)$\rightarrow$B is the operation which maps every a,b $\in$ B onto their join (a $\sqcup$ b)
5. 0 $\in$ B is the minimum of **B** and 1 $\in$ B is the maximum of **B**.

A **powerset Boolean algebra** is a structure <**pow**(A), $\subseteq$, $-$, $\cup$, $\cap$, Ø, A> where:
A is a set, $\subseteq$ is the subset relation on **pow**(A) and $-$, $\cup$, $\cap$ are the standard set theoretic operations on **pow**(A).

Take set {a,b,c}. Its powerset **pow**({a,b,c} is
{ Ø, {a}, {b}, {c}, {a,b}, {a,c}, {b,c}, {a,b,c} }
The following picture shows how these sets are related to each other by the subset relation $\subseteq$:




Let **A** = $<A, \sqsubseteq_A, \neg_A, \sqcap_A, \sqcup_A, 0_A, 1_A>$ and **B** = $<B, \sqsubseteq_B, \neg_B, \sqcap_B, \sqcup_B, 0_B, 1_B>$ be Boolean algebras.

> **A** and **B** are **isomorphic** iff there is an isomorphism from **A** into **B**.
>
> An **isomorphism** from **A** into **B** is a function $h:A \rightarrow B$ such that:
> 1. h is a **bijection**, a **one-one** function from A **onto** B.
> 2. h preserves the structure from **A** onto **B**:
>    a. for every $a \in A$: $h(\neg_A a) = \neg_B h(a)$
>    b. for every $a_1,a_2 \in A$: $h((a_1 \sqcap_A a_2)) = (h(a_1) \sqcap_B h(a_2))$
>    c. for every $a_1,a_2 \in A$: $h((a_1 \sqcup_A a_2)) = (h(a_1) \sqcup_B h(a_2))$
>    d. $h(1_A) = 1_B$
>    e. $h(0_A) = 0_B$

**Fact**: If h is an isomorphism from **A** into **B** then the inverse function $h^{-1}$ is an isomorphism from **B** into **A**.

Isomorphic Boolean algebras have the same Boolean structure.
They at most differ in the naming of the elements, but not in the Boolean relations between the elements.
The set theory that the theory of Boolean algebras is formulated in is extensional. This means that **A** and **B** are not **literally** the same Boolean algebra.
But mathematicians (and we) think of Boolean algebras as more intensional entities. With the mathematicians we will regard **A** and **B** as the same Boolean algebra.

In general, we do not distinguish between isomorphic structures:
if **A** happens to be an algebra of sets (like a powerset) and **B** an algebra of, say, functions, they are the same structure if they are isomorphic.
And that means that we are **free** to think of them as either sets or functions, whichever is more convenient. As we have seen, this double nature is one of the basic tenets of functional type theory.

Some theorems concerning Boolean algebras.

> **Representation Theorem:**
> Every complete atomic Boolean algebra is isomorphic to a powerset Boolean algebra.

**Proof sketch:**
Let **B** be a complete atomic Boolean algebra and let $ATOM_{\mathbf{B}}$ be the set of atoms in **B**.
For every $b \in B$ let $AT_b = \{a \in ATOM_{\mathbf{B}}: a \sqsubseteq b\}$
So $AT_b$ is the set of atoms below b, the set of b's atomic parts.
Define a function h: $B \to$ **pow**($ATOM_{\mathbf{B}}$) by:
for every $b \in B$: $h(b) = AT_b$.
What you can prove is that h is an isomorphism between **B** and
<**pow**($ATOM_{\mathbf{B}}$), $\subseteq, -, \cap, \cup, \emptyset$, $ATOM_{\mathbf{B}}$>.
So every complete atomic Boolean algebra is isomorphic to the powerset Boolean algebra of the powerset of its atoms.

This is not very difficult to prove, if you have first proved the following basic fact about complete atomic Boolean algebras:

**Fact**: In a complete atomic Boolean algebra every element is the join of its atomic parts:
for every $b \in B$: $b = \sqcup AT_b$.

The representation theorem tells us that the complete atomic Boolean algebras are just the powerset Boolean algebras. Since we have already seen that every finite lattice is complete and atomic, and hence so is every finite Boolean algebra, it follows that:

**Corrollaries:** The finite Boolean algebras are exactly the finite powerset Boolean algebras.

More generally:

1. Each powerset Boolean algebra has cardinality $2^n$, for some n (i.e. if $|A|= n$, $|pow(A)|=2^n$)
2. For each cardinality $2^n$ (for some n), there is exactly one powerset Boolean algebra, up to isomorphism.
3. The powerset Boolean algebras are up to isomorphism exactly the complete atomic Boolean algebras.

## Summary: the conceptual motivation for Boolean algebras

We are confronted with naturalistic domains for the interpretations of natural language expressions on which we assume a natural partial order.

The intuitive notions of sum and overlap are formalized via join and meet, where the intuitive notion of overlap and disjointness are defined with help of 0.

Distributivity formalizes the natural notion of partition: if you partition an object into two parts a and b, then any part of that object will be part of a, or of b, or it will be itself partitioned into two parts, one part of a and one part of b.

Complementation formalized the natural relation between proper parts and remainders: if a is a proper part of b, then there is a remainder part of b which does not overlap a.

Structures that satisfy these conditions are Boolean lattices: <B,⊑> or <B,⊔,⊓>.
Taking all the relevant notions as structure defining operations the same structures are Boolean algebras: <B,⊑,⊔,⊓,¬,0,1>

Boolean algebras are complete if ⊔ and ⊓ are defined for all subsets of B
Boolean algebras are atomic iff every element has an atomic part.
Complete atomic Boolean algebras are atomistic: every element is the sum of its atomic parts.

Powerset Boolean algebras are Boolean algebras of the form <**pow**(A),⊆,∪,∩,−,Ø,A>

**Representation Theorem:**
Every complete atomic Boolean algebra is isomorphic to a powerset Boolean algebra.

Reason:
If B is a complete atomic Boolean algebra it is atomistic, every element is the sum of its atomic parts.
You can prove that the function that maps every element of B onto the set of itse atomic parts is an isomorphism between <B,⊑,⊔,⊓,¬,0,1> and <**pow**($\mathrm{ATOM}_B$),⊆,∪,∩,−,Ø,$\mathrm{ATOM}_B$>.

From this the facts about cardinality of Boolean algebras follow:

-Each complete atomic Boolean algebra has cardinality $2^n$, for some n.
-For each cardinality $2^n$ (for some n), there is exactly one complete atomic Boolean algebra, up to isomorphism.

I will next discuss a theorem that gives a very deep insight in the structure of Boolean algebras.

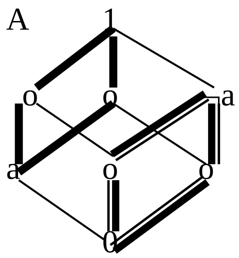
**The decomposition theorem**:
Every Boolean algebra which has an atom can be decomposed into two non-overlapping isomorphic Boolean algebras.

## Proof sketch:

Let B be a Boolean algebra and a an atom in B. Look at the following sets:
$B_a = \{b \in B: a \sqsubseteq b\}$, the set of all elements that a is part of.
$B_{\neg a} = \{b \in B: b \sqsubseteq \neg a\}$, the set of all parts of of ¬b.

It is not difficult to prove that: $B_a \cup B_{\neg a} = B$ and $B_a \cap B_{\neg a} = \emptyset$. This means that $B_a$ and $B_{\neg a}$ **partition** B.
Now, you can restrict the operations of B to $B_a$ and the result is a Boolean algebra $\mathbf{B_a}$ with a as 0. Similarly, $B_{\neg a}$ forms a Boolean algebra $\mathbf{B_{\neg a}}$ with ¬a as 1.
And you can prove that $\mathbf{B_a}$ and $\mathbf{B_{\neg a}}$ are isomorphic.

Example: pick atom a in A. Look at $B_a$ and $B_{\neg a}$:

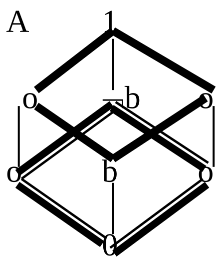


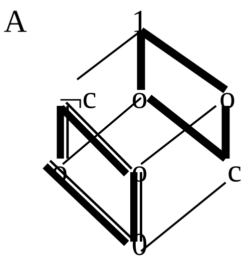

You see the two Boolean algebras sitting in the structure (each isomorphic to pow({a,b}).
It doesn't matter which atom you use here. In fact, there is a further fact:

**Fact:** If **B** is a Boolean algebra and a and b are atoms in **B** then $B_a$ and $B_b$ are isomorphic.

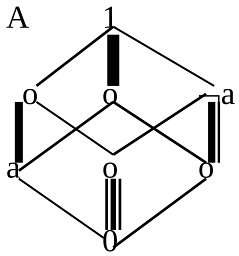
That means that in a given Boolean algebra, if you stand at any of its atom and you look up, the sky looks the same, you see the same structure (up to isomorphism).

So, we also get:




And it doesn't stop here. If $B_a$ has an atom, then by the same theorem, both $B_a$ and $B_{\neg a}$ themselves each decompose into two non-overlapping isomorphic Boolean algebras, so, all in all we have decomposed A into four isomorphic Boolean algebras:




The inverse of the decomposition theorem is the composition theorem:

## The Composition Theorem:

Let $\mathbf{B_1}$ and $\mathbf{B_2}$ be two Boolean algebras, that have no element in common, and that are isonmorphic. Then we can define a Boolean algebra on $B_1 \cup B_2$.

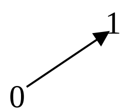
Proof sketch:
Note that $|B_1 + B_2| = |B_1| + |B_2|$, which is a power of 2, of $B_1$ and $B_2$ are Boolean algebras (so that is promising, in light of the facts stated above).
Let $h: B_1 \to B_2$ be the isomorphism. You define the order relation on $B_1 \cup B_2$ in terms of the orders $\sqsubseteq_{B1}$, $\sqsubseteq_{B2}$ and h.
Take the relation: $\sqsubseteq_{\mathbf{B1}} \cup \sqsubseteq_{\mathbf{B2}} \cup \{<a,h(a)>: a \in B_1\}$
The only problem with this relation is that it isn't transitive, so take its **transitive closure**, the result of **adding** all pairs <a,b> such that for some c: <a,c> and <c,b> stand in the above relation. The resulting relation is the partial order relation on $B_1 \cup B_2$. In the resulting partial order you can define the Boolean operations quite simply (eg. if $a \in B_1$, $\neg_{[B1 \cup B2]}(a) = h(\neg_{B1}(a))$; if $a \in B_2$, $\neg_{[B1 \cup B2]}(a) = h^{-1}(\neg_{B2}(a))$ )

This gives you a second procedure to generate all finite Boolean algebras (the first is as powersets), but this time one that actually gives you a **plotting algorithm**.

Start with two one element Boolean lattices: <{0},{<0,0>}> and <{1},{<1,1>}>
(These are really degenerate Boolean algebra, since 0 =1. We really start counting at 2.) And the isomorphism h = {<0,1>}. Then the composition theorem tells us how to form the two element Boolean algebra:

0

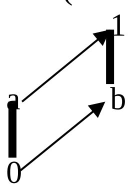
1-element Boolean algebra: a point. (We don’t count this as a Boolean algebra, actually)

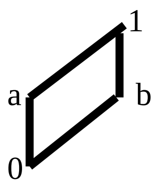




Two 1-element Boolean algebras give a 2-element Boolean algebra: a line.

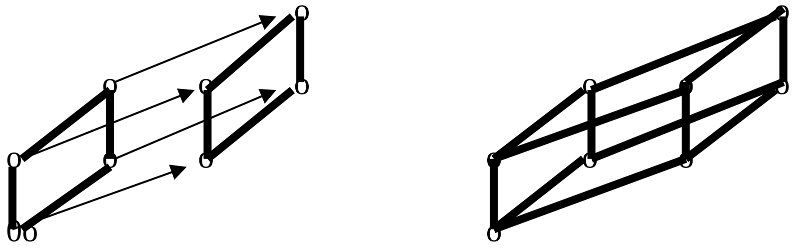
Take two disjoint two element Boolean algebra and an isomorphism: turn the arrow into a line (the transitive closure adds <0,1>), and you get the 4-element Boolean algebra:



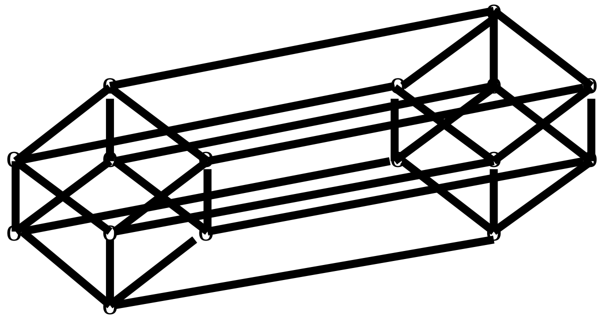

Two 2-element Boolean algebras give a 4 element Boolean algebra: a square.

Take two disjoint four element Boolean algebras and an isomorphism, form the eight element Boolean algebra:



Two 4-element Boolean algebras give an 8 element Boolean algebra: a cube.

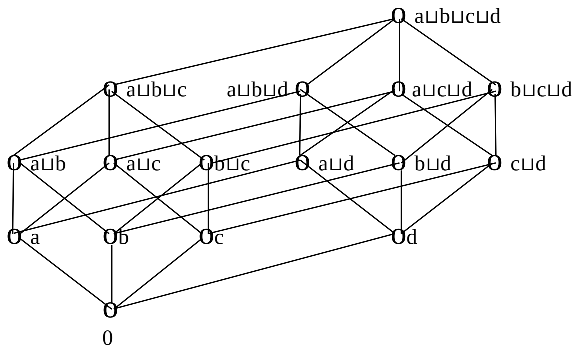
Take two disjoint eight element Boolean algebras and an isomorphism, form the sixteen element Boolean algebra.



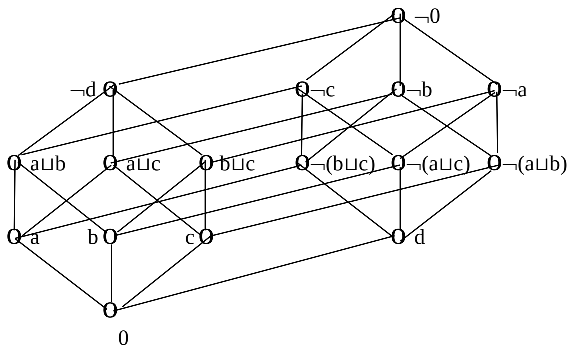
Two 8-element Boolean algebras form a 16 element Boolean algebra: a four dimensional cube.

Example: The 16 element Boolean algebra:

-with sum-specification:




-with complement specification:



**7.2. Boolean semantics for count nouns and mass nouns**
**[Adapted from Landman 2020]**

Classical logic:

-Basic domain of interpretation is an unordered set D of individuals
-One-place predicates are interpreted (at index w) as subsets of D.
-Nouns are standardly regarded as one-place predicates
-With that, the assumption made for simplicity is that all nouns are singular count nouns.

Boolean semantics:

-Replace the unordered domain of individuals by an ordered domain, viz. a complete Boolean algebra **B**.
-Link the count-mass distinction and the singular/plural distinction to atomicity:
    -Count objects are objects that are sums of atomic objects
    -The cardinality of a count object is the cardinality of the set of its atomic parts.
    -Singular objects are objects with cardinality 1, atoms.
    -Mass objects would be objects that are not built as sums of atoms.

Goes back to *mereology*, initiated by Leśniewski 1916, further developed by Leonard and Goodman 1940, and Goodman and Quine 1947.
As argued by Tarski 1935, mereologies are basically complete Boolean algebras restricted to objects (non-null elements).

Here: The variant of Boolean semantics that was initiated by Godehard Link in Link 1983 (and modified in Link 1984 to incorporate the analysis of the definite article by Sharvy 1980 and the operation of group formation).

-Link: The operation of *closure under sum* * is a *semantic operation* that maps
    semantically singular noun denotation X onto
    semantically plural noun denotation: $*X = \{b \in B: \exists Y \subseteq X: b = \sqcup Y\}$.

The assumption that this operation is available in the semantic system for natural language has made this operation a very fruitful tool for studying plurality cross-linguistically.

Topics that have been studied intensively are:
-the role of semantic pluralization in the semantics of counting phrases, distributivity, and count comparison;
-the relation between morphological plurality and semantic plurality;
-semantic plurality in languages without morphological plurality;
-the question of whether semantic pluralization is inclusive (*) or strict ($\oplus$);
-semantic plurality in the verbal domain and the role of semantic pluralization in the phenomena of distributivity and cumulativity;
- the strict event related plurality that seems to be involved in pluractionality morphemes.
And the list goes on.

## 7.3. Boolean semantics: basic notions

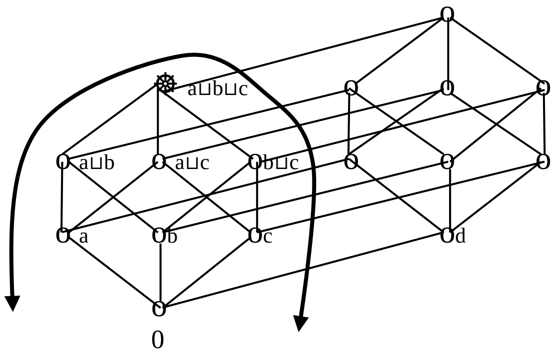
For clarity I repeat some notions.
-Boolean semantics assumes semantic interpretation domains which are complete Boolean algebras.
-It is useful to distinguish between elements of B and *objects* in B, where objects are non-null elements:

Let **B** be a Boolean algebra and $X \subseteq B$.
▹ $X^+$, the set of *objects* in X, is given by: $X^+ = X - \{0\}$

Following Grätzer 1978, I use half-closed interval notation for Boolean part sets:

Let $x \in B$, $X \subseteq B$
▹ The *Boolean part set* of x, **(**x**]**, is given by: **(**x**]** $= \{b \in B\colon b \sqsubseteq x\}$
▹ The *Boolean part set* of X, **(**X**]**, is given by: **(**X**]** = **(**$\sqcup$X**]**

The Boolean part set of a⊔b⊔c: **(**a⊔b⊔c**]**.

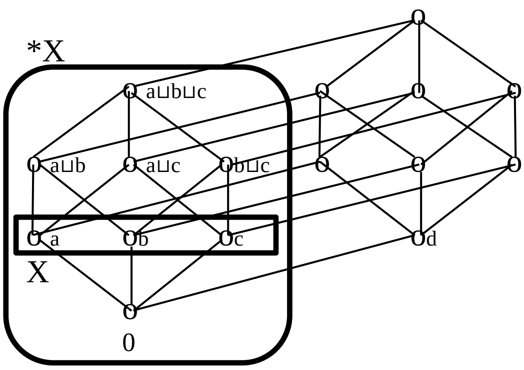



In Boolean semantics, notions of closure under sum and generation under sum are of central importance:

Let $X, Y \subseteq B$.
▹ The *closure under sum of* X, *X, is given by: $*X = \{b \in B\colon \text{for some } Y \subseteq X\colon b = \sqcup Y\}$

*X is the set of all sums of elements of X; *X contains ⊔Z, for each subset Z of X.

An example:




X = {a, b, c}
*X = {0, a, b, c, a⊔b, a⊔c, b⊔c, a⊔b⊔c}

Note: $0 \in *X$, because $\varnothing \subseteq X$ and $\sqcup\varnothing = 0$

If you want a notion of closure under sum that doesn't include 0, define, $^{+}X = (^{*}X)^{+}$.
If you want a notion of closure under sum that doesn't include X, define, $^{\oplus}X = {}^{+}X - X$.

The count domain is a complete atomic Boolean algebra.
The semantics of count nouns in a nutshell:

1. Counting is in terms of atomic parts: $|x| = |ATOM_x|$, where $ATOM_x = (x] \cap ATOM_b$
2. Hence atoms are objects of count 1, we call them singular objects.
3. Plural objects are sums of singular objects.
4. Singular nouns denote sets of atoms
   Semantic pluralization is closure under sum.

▹ A *count domain* is a complete atomic Boolean algebra **B**.

*Singular nouns* are interpreted as sets of atoms:

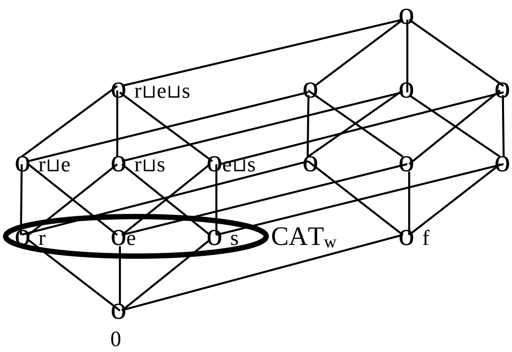
*cat* → $CAT_w$ with $CAT_w \subseteq ATOM_B$

*Plural nouns* are interpreted as the closure under sum of singular noun denotations:

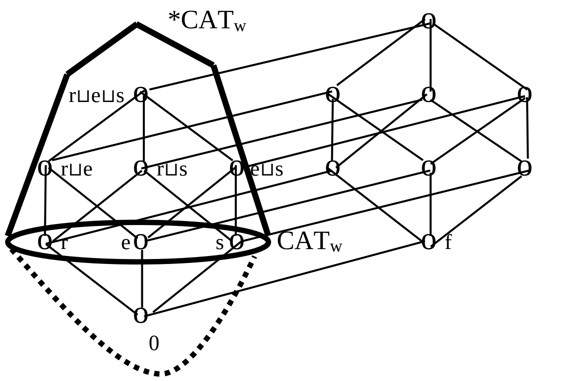
*cats* → $^{*}CAT_w$

Let $ATOM_B = \{r, e, s, f\}$ and let *Ronya* → r *Emma* → e *Shunra* → s *Fido* → f
We show $CAT_w$ and $^{*}CAT_w$.

*cat* → $CAT_w = \{r, e, s\} \subseteq ATOM_B$




*cats* → $^{*}CAT_w = \{0, r, e, s, r\sqcup e, r\sqcup s, e\sqcup s, r\sqcup e\sqcup s\}$




We assume that DP conjunction allows, besides the standard generalized conjunction interpretation based on $\wedge$, an interpretation as *sum conjunction*:

▹ *Sum conjunction*: *and* → $\lambda y \lambda x.\ x \sqcup y$ (at type <<e,t>,<e,t>>)

With this, we allow:

*ronya and emma and shunra* → $r \sqcup e \sqcup s$ $(\lambda y \lambda x. x \sqcup y(r,(\lambda y \lambda x. x \sqcup y(e,s))))$
*are cats* → $^{*}CAT_w$

Hence, (1a) gets interpreted as (1b):

(1) a. Ronya and Emma and Shunra are cats.
b. $*CAT_w(r \sqcup e \sqcup s)$
c. $CAT_w(r) \wedge CAT_w(e) \wedge CAT_w(s)$

**Lemma:** Given that $CAT_w \subseteq ATOM_B$, (1b) is equivalent to (1c).

**cats and dogs**

The proper analysis of conjunction of plural NPs is sum pairing:

*and* $\rightarrow \lambda Q \lambda P \lambda x.\exists x_1 \exists x_2[P(x_1) \wedge Q(x_2) \wedge x = x_1 \sqcup x_2]$

*cats and dogs* $\rightarrow$ $\textbf{and}_{NP} = \lambda x.\exists x_1 \exists x_2[*CAT_w(x_1) \wedge *DOG_w(x_2) \wedge x = x_1 \sqcup x_2]$

Let $CAT_w = \{r, e\}$ and $DOG_w = \{f, s\}$
Then

*cats and dogs* $\rightarrow$
$\{0, r, e, r\sqcup e, f, s, f\sqcup s, r\sqcup f, r\sqcup s, r\sqcup f\sqcup s, e\sqcup f, e\sqcup s, e\sqcup f\sqcup s, r\sqcup e\sqcup f, r\sqcup e\sqcup s, r\sqcup e\sqcup f\sqcup s\}$

Notice that this example shows the usefulness of the 0 element. Without 0 in the denotation of *X, you would only get:

*cats and dogs* $\rightarrow$
$\{r\sqcup f, r\sqcup s, r\sqcup f\sqcup s, e\sqcup f, e\sqcup s, e\sqcup f\sqcup s, r\sqcup e\sqcup f, r\sqcup e\sqcup s, r\sqcup e\sqcup f\sqcup s\}$

which is not enough.

Fact: if $P, Q \subseteq ATOM$ then:
$\textbf{and}_{NP}(*P, *Q) = *(P \cup Q)$

Landman 2004 extends this analysis to NPs of the form *Ronya, Shunra, and Emma and* ***maybe*** *Sasha*.

### 7.4. Counting in Boolean semantics

The analysis I present here is the one given in Landman 2004 (except for a little change in syntax). The guiding idea of this analysis is defended in a more general way in Landman 2010:

*Methodological guide*: The semantic interpretation and composition of numerical expressions takes place at the lowest available type.

For instance, analyses directly based on Barwise and Cooper 1982, van Benthem 1984 give as the semantics for *three*:

*three* → $\lambda Q\lambda P.|Q \cap P| \geq 3$ of type <<e,t>,<<e,t>,t>>

Landman 2000 gives:

*three* → $\lambda Q\lambda P.\exists x[Q(x) \wedge |x|=3 \wedge P(x)]$ of type <<e,t>,<<e,t>,t>>

Landman 2003 and Rothstein 2017 give:

*three* → $\lambda x.|x| = 3$ of type <e,t>

I propose with Landman 2004:

*Number*: *three* → 3 of type n (the type for numbers)

Clearly, the last proposal fits the methodological guide best.

The proposal has as immediate advantage that, unlike the other proposals, it directly suggest a natural compositional semantics for other number predicates like *at least three*, *at most three, exactly three, more than three,* ... The methodological guide instructs us to choose the simplest interpretation for expressions *at least*, *at most*, *exactly*, *more than*: on the simplest interpretation they denote the obvious *relations between numbers*:

*Number relations*: *at least* → $\geq$ *at most* → $\leq$ *more than* → $>$ *exactly* → $=$
of type <n,<n,t>> (the type of relations between numbers)

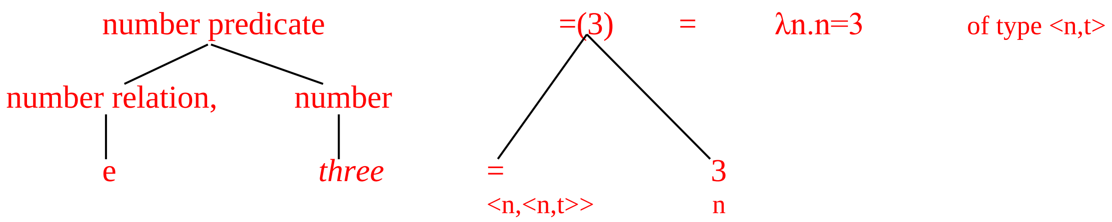
Again, the methodological guide tells us that the simplest way of combining a number relation with a number is application, resulting in a number predicate. Thus:

*Number relation + Number → Number predicate = (Number relation(Number))*

*Number predicates*: *at least three* → $\geq(3)$ of type <n,t>
(the type of number predicates)
$\geq(3) = \lambda n.n \geq 3$
the set of numbers bigger or equal to 3

Because the analysis of *at least three* will also form the input for the analysis of measure phrases like *at least three liters*, in principle the domain of type n is $\mathbb{R}^+$, the set of non-negative real numbers.
So $\lambda n.n \geq 3$ is the set of *real* numbers from 3 up.
However, as we will see, in a count context the number predicate composes with the cardinality function, which will bring the domain down to natural numbers, so that in that context $\lambda n.n \geq 3$ will be the set of natural numbers from 3 up.

*At least three*, so far, denotes the number predicate $\lambda n.n \geq 3$. I assume that also unmodified number expression *three* must receive an interpretation at the type of number predicates $<n,t>$.
With Landman 2004 I assume for simplicity a null number relation node, and I assume that the default interpretation for the null number relation is the identity relation between numbers, =:



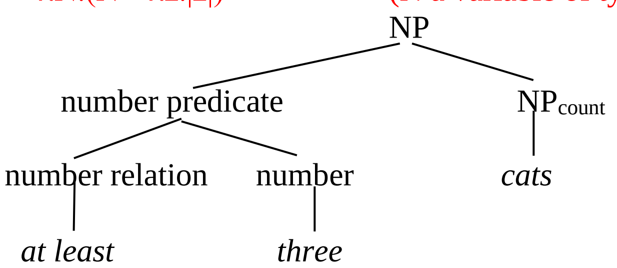

A number predicate like *at least three* combines with an NP like *cats*.
The simplest assumption (modified only slightly below) is that, like intersective adjectives, they combine via intersection.
Since the type of the interpretation of the NP *cats* is $<e,t>$, a predicate of individuals, the number predicate must receive an interpretation at that type as well.
It does so via *function composition* with the cardinality function:

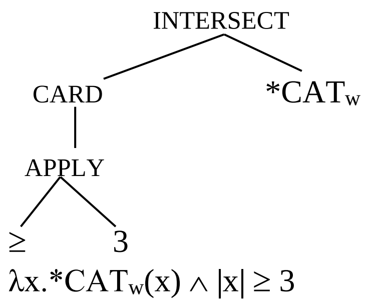
*at least three*$_{<n,t>}$ + **card**$_{<e,n>}$ $\rightarrow$ *at least three*$_{<e,t>}$
$\lambda n.n \geq 3 \quad \circ \quad \lambda z.|z| \quad = \quad \lambda x.|x| \geq 3$

$\lambda n.n \geq 3 \circ \lambda z.|z| \quad =$
$\lambda x.[\lambda n.n \geq 3(\lambda z.|z|(x))] \quad =$
$\lambda x.[\lambda n.n \geq 3(|x|)] \quad =$
$\lambda x.|x| \geq 3$

I assume the structure below and assume a type shifting rule CARD:

▹ CARD: $<n,t> \rightarrow <e,t>$
CARD $= \lambda N.(N \circ \lambda z.|z|)$ (N a variable of type $<n,t>$)



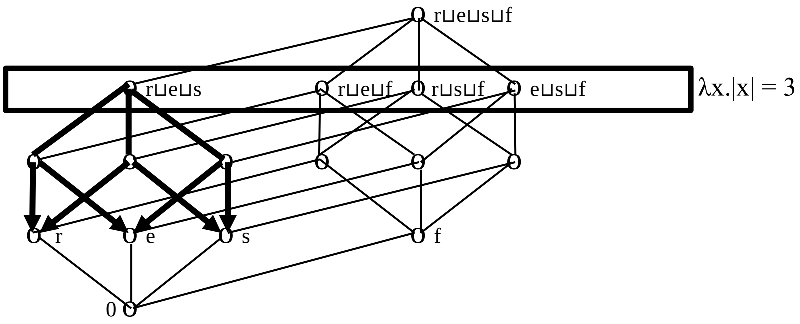




On the semantics given, we derive a number predicate *three* with, after type shifting with CARD, the following interpretation:

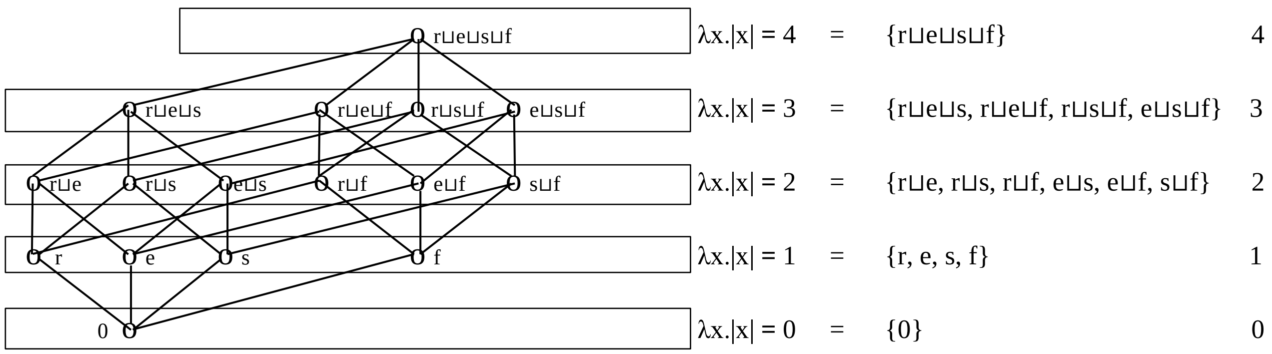
*three* $\rightarrow \lambda x.|x|=3$

$\lambda x.|x|=3$ is the set of all elements $b \in B$ such that $|ATOM_b|=3$,
i.e. $|(b] \cap ATOM_B| = 3$,
the set of all objects with three atomic parts.

For example, in the picture below r⊔e⊔s ∈ λx.|x|=3.
The set of atomic parts of r⊔e⊔s is {r, e, s}, which has indeed three elements.
On the other hand r⊔e⊔s⊔f ∉ λx.|x|=3, and nor is r⊔e.
The first has four atomic parts, the second only two.




In general: λx.|x|=n denotes the set of elements of **B** at height n in B:

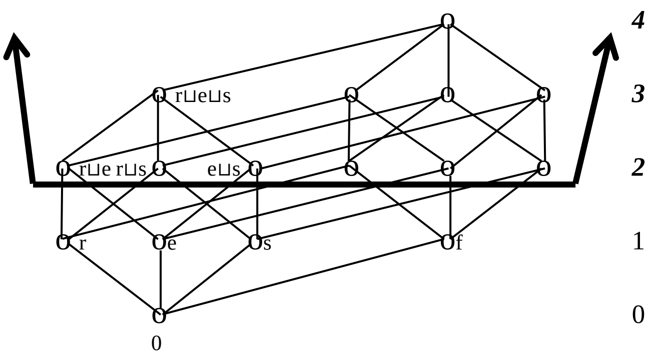


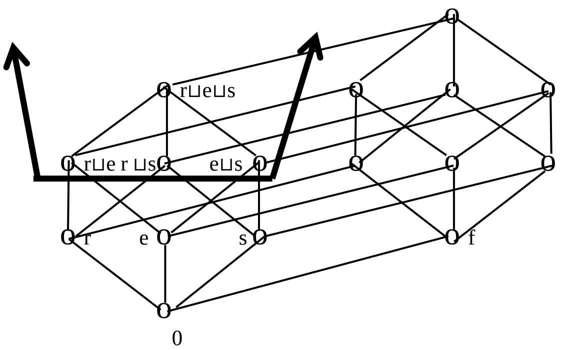

The compositional analysis of number phrases in Boolean semantics brings out the fact that the monotonicity properties of numerical noun phrases derive from the monotonicity properties of the number relation, and it makes the monotonicy properties beautifully visible.

*Closed Up:*

*At least two* → λx.|x|≥2

*At least two cats* → λx.*CAT$_{w,t}$(x) ∧ |x|≥2

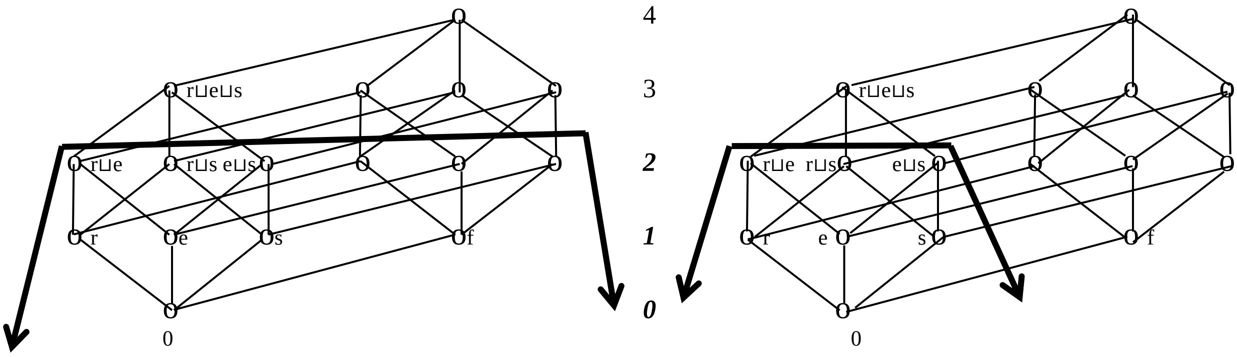


*Closed Down:*

*At most two* → $\lambda x.|x|\leq 2$

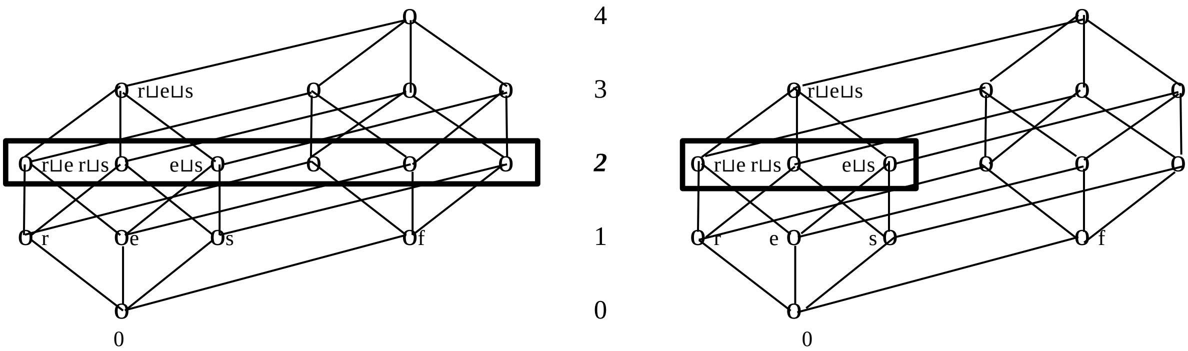
*At most two cats* → $\lambda x.{}^*CAT_{w,t}(x) \wedge |x|\leq 2$




*Closed Neither Up Nor Down:*

*Exactly two* → $\lambda x.|x|=2$

*Exactly two cats* → $\lambda x.{}^*CAT_{w,t}(x) \wedge |x|=2$




In the modifier context, it will be useful to make the presuppositional effect of CARD explicit.
We do that most easily by letting CARD derive not a predicate at type <e,t>, but a *presuppositional intersective modifier* at type <<e,t>,<e,t>>:

*Presuppositional cardinality shift:*

$$\triangleright\text{CARD} = \lambda N\, \lambda P. \begin{cases} P \cap (N \circ \lambda x.|x|) & \text{if } P \text{ is } \textit{count} \\ \bot & \text{otherwise} \end{cases}$$

We apply CARD to the number predicate interpretation of *at least three*:

*at least three* → $\lambda n.n \geq 3$

and we get:

$$\text{CARD}(\lambda n.n \geq 3) = \lambda P. \begin{cases} \lambda x.P(x) \wedge |x|\geq 3 & \text{if } P \text{ is } \textit{count} \\ \bot & \text{otherwise} \end{cases}$$

We now have a modifier interpretation for *at least three* which is undefined if it combines with a head NP whose interpretation is not count.

When defined, as in the case of $*CAT_w$, the result of applying CARD($\lambda n.n \geq 3$) to $*CAT_w$ denotes, as before, the set of sums of cats with three atomic parts.

What we haven't defined here is what it means for a set of type <e,t> to be count.
This can be defined in terms of atomicity:

*Count sets*: Let B be a complete Boolean algebra and $P \subseteq B$.
▻ P is *count* iff if $P^+ \neq Ø$ then **(**P**]** is a complete *atomic* Boolean algebra.

If we want to use this to account for the felicity of number predicates with count nouns (✓*at least three cats*) and the infelicity of number predicates with mass nouns (#*at least three mud(s)*), we need define what it means for a *noun* to be a count noun.

This can be done in terms of intensions. We are here only interested in intensions of type <s,<e,t>>, functions from set of indices W to subsets of B.

Let P: W → **pow**(B) be an intension.

▻ *Count intensions*: P is *count* iff for every $w \in W$: $P_w$ is count.

And we make the obvious assumption:

*Count noun phrases*: Count noun phrases are interpreted as *count* intensions.

With this we can actually formulate an intensional version of the above shifting rule CARD:

Let *P* be a variable over intensions.

$$\text{CARD}(\lambda n.n \geq 3) = \quad \lambda P. \begin{cases} \lambda x.P_w(x) \wedge |x| \geq 3 & \text{if } P \text{ is a } count \text{ intension} \\ \bot & \text{otherwise} \end{cases}$$

This means that we apply CARD($\lambda n.n \geq 3$) to the intension of *cats*: $\lambda w.*CAT_w$.
This is indeed a count intension, hence we derive *at least three cats* with the standard interpretation.
If we apply CARD($\lambda n.n \geq 3$) to the intension of *mud* , $\lambda w.MUD_w$, we assume that this intension is not count, and no felicitous interpretation is derived.

In Boolean semantics counting makes reference to $ATOM_B$.
Since we let the denotation of a singular predicate like *cat* be a set of atoms, the objects in this denotation, singular cats, are by definition objects of cardinality one.
Objects in the plural denotation *cats* are counted in terms of their atomic parts.

### 7.5. Sharvy's definiteness operation and the pragmatics of the null element

Link 1983 analyzes the definite article *the* as the sum operation $\sqcup$.
Link 1984 follows Sharvy 1980 in analyzing *the* as a presuppositional sum operation:

▹ *Definite article*: [Sharvy 1980]
*the* → σ

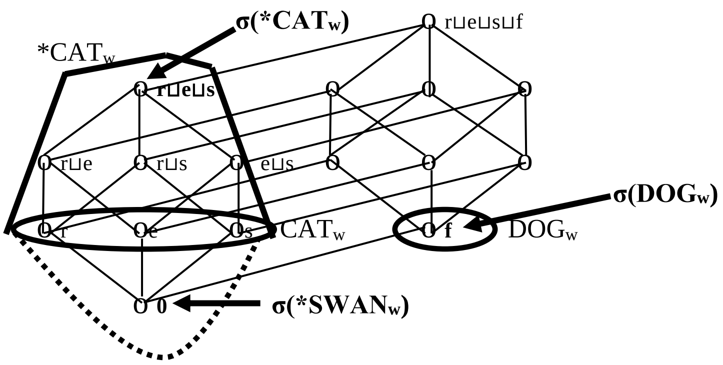
$$\sigma = \lambda P. \begin{cases} \sqcup P & \text{if } \sqcup P \in P \\ \bot & \text{otherwise} \end{cases}$$

σ is a presuppositional maximalization operation:
if P contains maximal element $\sqcup P$, then σ(P) is that element, otherwise it is undefined.
This means that σ(P) *presupposes* that P contains $\sqcup P$.

Assume the following noun denotations:

*cat* → $CAT_w$ = {r, e, s}
*dog* → $DOG_w$ = {f}
*swan* → $SWAN_w$ = Ø

The picture shows $\sigma(*CAT_w)$, $\sigma(DOG_w)$, $\sigma(SWAN_w)$ and $\sigma(*SWAN_w)$:




$SWAN_w$ = Ø

$\sigma(SWAN_w) = \bot$

More in detail, we find the following:

denotation:

*the cat* → $\sigma(CAT_w)$ $\bot$
$\sigma(CAT_w) = \bot$ because $\sqcup(CAT_w) = r\sqcup e\sqcup s$ and $r\sqcup e\sqcup s \notin CAT_w$

*the swan* → $\sigma(SWAN_w)$ $\bot$
$\sigma(SWAN_w) = \bot$ because $SWAN_w$ = Ø and $\sqcup$(Ø) = 0 and 0 ∉ Ø

*the dog* → $\sigma(DOG_w)$ f
$\sigma(DOG_w)$ = f because $\sqcup\{f\}$ = f and f ∈ $DOG_w$

This shows that Sharvy's σ operation generalizes Russell's iota operation ι:
for singular noun denotations σ makes the same predictions as ι.
For plural predicates, σ picks out the maximal element $\sqcup P$ of P, if defined:

denotation:

*the cats* $\rightarrow \sigma(*CAT_w)$ r⊔e⊔s

$\sqcup(*CAT_w)$ = r⊔e⊔s and r⊔e⊔s $\in *CAT_w$

*the three cats* $\rightarrow \sigma(\lambda x.*CAT_w(x) \wedge |x|=3)$ r⊔e⊔s

$\sqcup(\lambda x.*CAT_w(x) \wedge |x|=3)$ = r⊔e⊔s and r⊔e⊔s $\in \lambda x.*CAT_w(x) \wedge |x|=3$

*the more than two cats* $\rightarrow \sigma(\lambda x.*CAT_w(x) \wedge |x|>2)$ r⊔e⊔s

$\sqcup(\lambda x.*CAT_w(x) \wedge |x|>2)$ = r⊔e⊔s and r⊔e⊔s $\in \lambda x.*CAT_w(x) \wedge |x|>2$

*the less than four cats* $\rightarrow \sigma(\lambda x.*CAT_w(x) \wedge |x|<4)$ r⊔e⊔s

$\sqcup(\lambda x.*CAT_w(x) \wedge |x|<4)$ = r⊔e⊔s and r⊔e⊔s $\in \lambda x.*CAT_w(x) \wedge |x|<4$

*the two cats* $\rightarrow \sigma(\lambda x.*CAT_w(x) \wedge |x|=2)$ $\perp$

$\sqcup(\lambda x.*CAT_w(x) \wedge |x|=2)$ = r⊔e⊔s and r⊔e⊔s $\notin \lambda x.*CAT_w(x) \wedge |x|=2$

*The cats, the three cats, the more than two cats, the less than four cats* all denote r⊔e⊔s;
*the two cats* is undefined.

This theory makes an interesting distinction between definites that are undefined ($\perp$) and definites that denote 0: The distinction is shown for NPs based on singular noun *swan*, with the given assumption that $SWAN_w = \emptyset$:

denotation:

*the swan* $\rightarrow \sigma(SWAN_w)$ $\perp$

$\sigma(SWAN_w) = \perp$ because $SWAN_w = \emptyset$ and $\sqcup(\emptyset) = 0$ and $0 \notin \emptyset$

*the more than three swans* $\rightarrow \sigma(\lambda x.*SWAN_w(x) \wedge |x|>3)$ $\perp$

$\sigma(\lambda x.*SWAN_w(x) \wedge |x|>3) = \perp$ because $\lambda x.*SWAN_w(x) \wedge |x|>3 = \emptyset$ and $\sqcup(\emptyset) = 0$ and $0 \notin \emptyset$

*the swans* $\rightarrow \sigma(*SWAN_w)$ 0

$\sigma(*SWAN_w) = 0$ because $*SWAN_w = *\emptyset = \{0\}$ and $\sqcup(\{0\})=0$ and $0 \in \{0\}$

*the less than three swans* $\rightarrow \sigma(\lambda x.*SWAN_w(x) \wedge |x|<3)$ 0

$\sigma(\lambda x.*SWAN_w(x) \wedge |x|<3) = 0$, because $\lambda x.*SWAN_w(x) \wedge |x|<3 = \{0\}$ and $\sqcup(\{0\})=0$ and $0 \in \{0\}$

I discussed this contrast in Landman 2004 and Landman 2010.
Frege's universal quantifier: the non-emptiness, non-singleton effect here is an implicature and not a presupposition:

[I ran for some years a crackpot lottery and stand in court. I know that I shouldn't commit perjury. But I think I am better at Gricean pragmatics than the judge is, so I say:]

(4) a. Your honor, I swear that every person who, in the course of last year, presented me with a winning lottery ticket, has gotten his prize.

[I add, *sotte voce*, to you:]

b. Fortunately I was away all year on a polar expedition.

(4b) tells you (but not the judge) that the denotation of the NP *person who, in the course of last year, presented me with a winning lottery ticket* is empty. If the non-emptiness condition were a presupposition, the continuation (4b) should be infelicitous, because it directly contradicts the presupposion. But (4b) is not infelicitous, it cancels the non-emptiness implicature, and makes statement (4a) *trivially true*.

This means, of course, that (4a) violates the Gricean maxim of Quantity, because it doesn't give any information. But that is exactly my intention: I make a statement that is trivially true (no perjury), hoping that the judge (using standard Gricean reasoning) believes that it *is* true (Quality), but non-trivially so (Quantity). So I am trying to mislead the judge without making a false statement.

We now look at felicity versus triviality of definite DPs in the same courtroom context as in example (4). Imagine that I said instead of (4) one of the statements in (5):

(5) a. Your honor, I swear that *the one person* who, in the course of last year, presented me with a winning lottery ticket, has gotten his prize.
[*sotte voce*, to you:] b. #Fortunately I was away all year on a polar expedition.
b. Your honor, I swear that *the five persons* who, in the course of last year, presented me with a winning lottery ticket, have gotten their prize.
[*sotte voce*, to you:] b. #Fortunately I was away all year on a polar expedition.
c. Your honor, I swear that *the more than thirty persons* who, in the course of last year, presented me with a winning lottery ticket, have gotten their prize.
[*sotte voce*, to you:] b. #Fortunately I was away all year on a polar expedition.

In all these cases the continuation is infelicitous. Why? Because the examples in (5) *presuppose* that respectively one/ five/more than thirty persons came to me with a winning lottery ticket. And the continuation *denies* that. That is as good as a contradiction: the continuation conflicts with the presupposition.

We compare these cases with the cases in (6). Now imagine that I had said any of the statements in (6):

(6) a. Your honor, I swear that *the persons* who, in the course of last year, presented me with a winning lottery ticket, have gotten their prize.
[*sotte voce*, to you:] b. ✓Fortunately I was away all year on a polar expedition.
b. Your honor, the books ought to tell you how many people came to me last year to claim their prize. I am sure it was less than five. But I swear to you, your honor, that *the less than five persons* who, in the course of last year, presented me with a winning lottery ticket, have gotten their prize.
[*sotte voce*, to you:] b. ✓Fortunately I was away all year on a polar expedition.

The cases in (6) pattern with *every* in (4): the continuation is felicitous, which indicates that the non-emptiness claim is a cancellable implicature rather than a presupposition, and hence that the cases in (6) are quantity violations, rather than perjury.

Not all native English speakers that I have consulted are completely happy with DPs where the numerical is complex, in particular cases like *the at least ten persons who…* (although searching for cases like that on the web gives a surprisingly rich harvest). But even they agree that there is a *robust contrast* between the cases in (5) and in (6), and that is what is important here.

The cases in (5) are explained by the standard assumption concerning undefinedness and presupposition failure:

▹ *Definiteness:* If $\sigma(P_w) = \bot$ then $\varphi_w(\sigma(P_w))$ is *infelicitous*, due to presupposition failure.

The cases in (6) are explained by pragmatic manipulation of the null element:

▹ *Triviality:* If $\sigma(P_w) = 0$, then $\varphi_w(\sigma(P_w))$ is *trivial*, either trivially true or trivially false.

The assumption in Landman 2011 is that whereas the inclusion/exclusion of 0 in the denotation of NP denotations in languages like English is determined by and large by the compositional semantics, the inclusion/exclusion of 0 in the denotation of *verbal* predicates can be manipulated by pragmatics (just as semantic plurality for verbal predicates is not strictly fixed by compositional semantics: morphological number in the verbal domain is linked to agreement, not to semantic plurality, e.g. Landman 2000).

In the cases in (6), I make the statement $\varphi_w(\sigma(P_w))$ in front of the judge, where $\sigma(P_w) = 0$. The assumption that I *do* obey Quality leads to the assumption that $0 \in \varphi_w$, and if $0 \notin \varphi_w$ it leads to accommodating 0 in $\varphi_w$, shifting from $\varphi_w$ to $\varphi_w \cup \{0\}$. With that, the statement $\varphi_w(\sigma(P_w))$ is trivially true.

### 7.6. Count comparison

I am concerned here with the *comparative* reading of *most* in (9a),
the reading on which (9a) means that more Dutch persons voted for Conchita than didn't.

(9) a. Most *Dutch persons* voted for Conchita.
b. #Most *Dutch person* voted for Conchita.

There are other interpretation possibilities (see e.g. Hackl 2009), but as far as the discussion here is concerned these have the same properties as the reading I am concerned with, so I use the comparative reading as a stand-in for all of them.

In classical Generalized Quantifier Theory (e.g. van Benthem 1984) the generalized quantifier interpretation of *most* in (9a) compares the cardinality of the intersection of the interpretations of the NP *Dutch persons* and the VP *voted for Conchita* with their difference:

▹ $most_{[count],GQT} \rightarrow \lambda Q\lambda P.\ |Q \cap P| > |Q - P|$

A puzzle for Generalized Quantifier Theory is why the count noun complement of *most* has to be plural,
as is shown in (9b).
In Boolean semantics, the interpretation of *Dutch persons* as a set closed under sum makes the above analysis unavailable.
Here the natural analysis is in terms of *relative complement*:

*Dutch person* $\rightarrow$ $DUTCH_w$ $\subseteq ATOM_B$
*voted for Conchita* $\rightarrow$ $\lambda x.VOTE_w(x,\text{CONCHITA})$ $\subseteq ATOM_B$

You take $\sqcup(^*DUTCH_w)$, the sum of all Dutch persons,
and $\sqcup(\lambda x.VOTE_w(x, \text{CONCHITA}))$, the sum of all persons that voted for Conchita.

The meet of these two is:

$\sqcup(*DUTCH_w) \sqcap \sqcup(\lambda x.VOTE_w(x, \text{CONCHITA}))$,
the sum of all Dutch persons who voted for Conchita.

The relative complement of this, relative to $\sqcup(*DUTCH_w)$ is:

$\sqcup(*DUTCH_w) - \sqcup(\lambda x.VOTE_w(x, \text{CONCHITA}))$,
the sum of all Dutch persons that have no part in common
with the sum of Dutch people that voted for Conchita,
which *de facto* is the sum of all Dutch persons who didn't vote for Conchita.

And the semantics of $most_{[count]}$ says that the cardinality of the first is bigger than the cardinality of the second, as in ($9a_1$):

(9) $a_1$. $|\sqcup(*DUTCH_w) \sqcap \sqcup(\lambda x.VOTE_w(x,\text{CONCHITA}))| >$
$|\sqcup(*DUTCH_w) - \sqcup(\lambda x.VOTE_w(x,\text{CONCHITA}))|$

This gives as the semantics for $most_{[count]}$:

$\triangleright\ most_{[count], preliminary} \rightarrow\ \lambda Q\lambda P.\ |\sqcup Q \sqcap \sqcup P|\ >\ |\sqcup Q - \sqcup P|$

This analysis has the same problem as the Generalized Quantifier analysis:
it gives no reason why the NP complement of *most* should be plural.

If we look around at other languages, we see that Romance languages use a different syntax which motivates the choice of the semantic plural directly:
Spanish, for instance, uses a partitive construction with a definite plural DP as complement:

(10) La mayoría de los holandeses
The majority of the Dutch persons

We can get basically the same effect in English, without imposing a partitive structure, by building definiteness of the nominal argument into the semantics of *most*:

$\triangleright\ most_{[count]} \rightarrow\ \lambda Q\lambda P.\ |\sigma(Q) \sqcap \sqcup P|\ >\ |\sigma(Q) - \sqcup P|$

The problem was: the semantics uses $\sqcup Q$. $\sqcup(DUTCH_w) = \sqcup(*DUTCH_w)$,
so why does it matter whether $Q = DUTCH_w$ or $Q = *DUTCH$?
We now replace $\sqcup Q$ by $\sigma Q$, and suddenly the difference matters:

in a context of comparison $\sigma(DUTCH_w)$ is not well defined,
since, when defined, it denotes a single atom, and cannot form a context for relative complement.

$\sigma(*DUTCH_w)$ on the other hand is perfectly suited to define a context for relative complement.

So the rationale for plurality is that for relative complement to be properly defined,
you need a plural $\sigma(*DUTCH_w)$.

Another problem to be solved is that so far ($9a_1$) does not seem to give the right truth conditions for (9a):
even if we fix the time to May 10, 2014, most Dutch persons obviously didn't vote at all in any voting going on that day, so without contextual restriction ($9a_1$) is false:

$$|\sigma(^*\text{DUTCH}_w) - \sqcup(\lambda x.\text{VOTE}_w(x,\text{CONCHITA}))| > |\sigma(^*\text{DUTCH}_w) \sqcup \sqcup(\lambda x.\text{VOTE}_w(x,\text{CONCHITA}))|,$$

Clearly, the semantics of *most* requires serious contextual restriction.
In the appropriate context for (9a) the interpretation of *Dutch person* is restricted to that of *Dutch person that voted in the final of the Eurovision Song Contest 2014*.

Let **c** be a context variable of type <s,<e,t>>. We assume that: $\mathbf{c}_w \subseteq \text{ATOM}_\mathbf{B}$.
In our case $\mathbf{c}_w = \text{VOTER}_{\text{ESC-2014},w}$, the set of singular individuals that voted in ESC 2014.
Clearly, then, we want to derive:

(9) $a_2$. $|\sigma(^*(\text{DUTCH}_w \cap \mathbf{c}_w)) \sqcap \sqcup(\lambda x.\text{VOTE}_w(x,\text{CONCHITA}))| >$
$|\sigma(^*(\text{DUTCH}_w \cap \mathbf{c}_w)) - \sqcup(\lambda x.\text{VOTE}_w(x,\text{CONCHITA}))|$

We can make contextual restriction a condition on the process of combining $most_{[count]}$ with its complement NP:

$most_{[count]} +_\mathbf{c} \text{NP}_{[plur]} \rightarrow [_{DP}\ most_{[count]}\ \text{NP}_{[plur]}]$

*constraint:* Restrict the interpretation of $\text{NP}_{[plur]}$ with **c**:

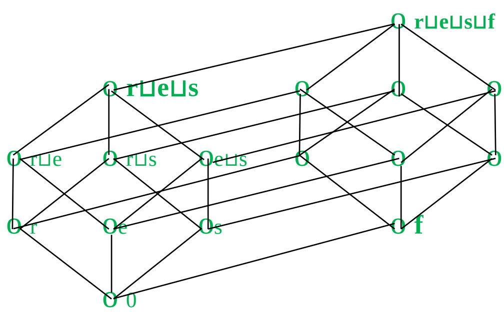
*Dutch persons* $\rightarrow_\mathbf{c}\ ^*(\text{DUTCH}_w \cap \mathbf{c}_w) = \ ^*(\text{DUTCH}_w \cap \text{VOTER}_{\text{ESC-2014},w})$

Alternatively, we can make the contextual restriction part of the semantics of *most*.
For instance, for many cases the following will do:

▹ $most_{[count],\mathbf{c}} \rightarrow \lambda Q\lambda P.\ |\sigma(^*(Q \cap \mathbf{c}_w)) \sqcap \sqcup P| > |\sigma(^*(Q \cap \mathbf{c}_w)) - \sqcup P|$

This works in our example, because $^*(^*\text{DUTCH}_w \cap \mathbf{c}_w) = {}^*(\text{DUTCH}_w \cap \mathbf{c}_w)$.

A situation in which ($9a_2$) is true.




r⊔e⊔s⊔f is the sum of all voters
r⊔e⊔s is the sum of all Conchita voters

r⊔e⊔s⊔f – r⊔e⊔s = f

|r⊔e⊔s| = 3 |f| = 1
3 > 1

The above semantics of *most*$_{[count]}$ establishes the sum of the Dutch persons who voted at the $ESC_{2014}$, and within the part set of that sum, the semantics compares the cardinality of the sum of the persons that voted for Conchita with the cardinality of the sum of the persons that didn't.
For each of these sums, we look at the set of its atomic parts, and we compare the sizes of those two sets.
Thus, count-comparison is in terms of *atomic parts*, which are parts that are atoms in B.

### 7.7. The distributive operator

We now look at examples like (13):

(13) a. *The three cats* ate half a can of tuna.
b. *The three cats* ate half a can of tuna *each*.

In (13a) it is undetermined whether the cats ate half a can of tuna together, or whether each of them ate that much tuna.
(13b) contains distributor *each* and allows only the second reading. On the distributive reading it is not the sum of three cats that has the property of eating half a can of tuna, but each of the individual cats making up that sum.

*the three cats* $\rightarrow \sigma(*\lambda x.*CAT_w(x) \wedge |x|=3)$

In Boolean semantics, the denotation of *the three cats* in w is the sum of its atomic parts, and provably the set of its atomic parts in w is a set of singular cats. i.e. they are themselves in the denotation of $CAT_w$.
This means that Boolean semantics is ideally suited to deal with distributivity.
Because of the atomistic structure of the domain, every count sum keeps track of its atomic parts, and that is exactly to which predicates distribute on the distributive reading: applying a distributive predicate to a sum *is* applying the predicate not to the sum itself but to its atomic parts.

Link 1983 proposes that *each* in examples like (13b) is interpreted as a distributive operator $^D$ that operates at the VP level:

▹ $^D = \lambda P \lambda x.\ ATOM_x \subseteq P$

**Lemma:** For all $P \subseteq B$: $^DP = *(P \cap ATOM_B)$

**Proof**: 1. Assume $x \in {}^DP$. Then $ATOM_x \subseteq P$. Then $P \cap ATOM_x = ATOM_x$. Since $x = \sqcup ATOM_x$, $x \in *ATOM_x$.
Hence $x \in *(P \cap ATOM_x)$, and hence $x \in *(P \cap ATOM_B)$.
2. Assume $x \in *(P \cap ATOM_B)$. Then for some $A \subseteq (P \cap ATOM_B)$: $x = \sqcup A$. By atomisticity, $A = ATOM_x$.
Hence $ATOM_x \subseteq P$, and hence $x \in {}^DP$.

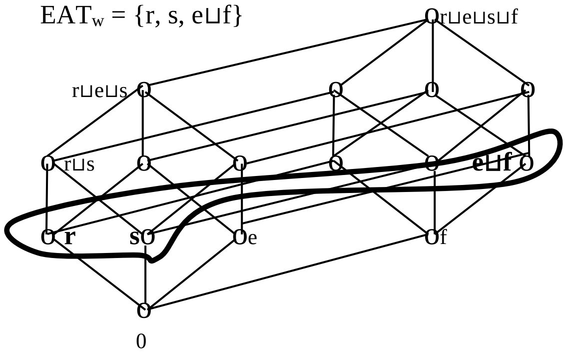
*cat* → $CAT_w = \{r, e, s\}$
*the three cats* → $\sigma(\lambda x.{*}CAT_w(x) \wedge |x|=3) = r \sqcup e \sqcup s$
*eat half a can of tuna* → $EAT_w$ with $EAT_w \subseteq B$
*eat half a can of tune each* → ${}^{D}EAT_w = \lambda x.\ ATOM_x \subseteq EAT_w$

(13b) is interpreted as (13c):

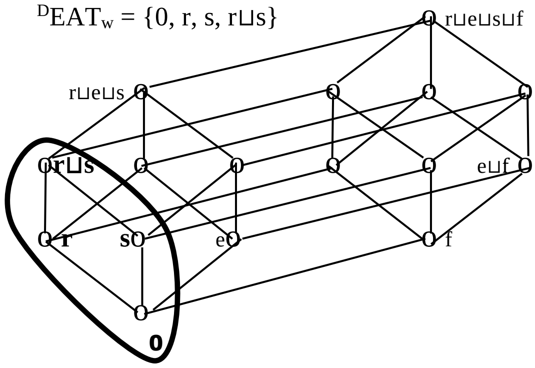
(13) c. $\lambda x.\ ATOM_x \subseteq EAT_w\ (r \sqcup e \sqcup s)$ =
$ATOM_{r \sqcup e \sqcup s} \subseteq EAT_w$ =
$EAT_w(r) \wedge EAT_w(e) \wedge EAT_w(s)$

The relation between $EAT_w$ and ${}^{D}EAT_w$ is:

$EAT_w = \{r, s, e \sqcup f\}$




${}^{D}EAT_w = \{0, r, s, r \sqcup s\}$




The piucture shows a situation where $EAT_w = \{r, s, e \sqcup f\}$:
say, Ronya eats half a can of tuna, Shunra eats half a can of tuna, and Emma and Fido eat half a can of tuna together.
Then (14a) and (14b) are not true, not on the collective reading and not on the distributive reading, but (14c) is true on the distributive reading:

(14) a. The animals ate half a can of tuna.
$r \sqcup e \sqcup s \sqcup f \notin EAT_w$ and $ATOM_{r \sqcup e \sqcup s \sqcup f} \not\subseteq EAT_w$
b. The cats ate half a can of tuna.
$r \sqcup e \sqcup s \notin EAT_w$ and $ATOM_{r \sqcup e \sqcup s} \not\subseteq EAT_w$
c. Ronya and Shunra ate half a can of tuna
$r \sqcup e \notin EAT_w$ but $ATOM_{r \sqcup s} \subseteq EAT_w$

We see that the semantics of distributive adverbial *each*, and more generally the distributive operator, make reference to atoms in B.

**7.8. Sums and groups.**

We assume so far that there are predicates of individuals

*purr* → $PURR_w \subseteq ATOM$

There are predicates of pluralities:

*meet on the roof* → $MEET_w \subseteq B - ATOM$

There are mixed predicates:

*carry the ball upstairs* → $CARRY_w \subseteq B$

Of course, I am ignoring here internal structure.
We can now analyze *three cats as*:

*three cats* → $\lambda P.\exists x[{*}CAT_w(x) \wedge |x|=3 \wedge P(x)]$

The set of properties that some sum of three cats has

Then we get:

*three cats purr* → $APPLY[PURR_w, \lambda P.\exists x[{*}CAT_w(x) \wedge |x|=3 \wedge P(x)]]$

There is a sortal mismatch here: x ranges over pluralities, but $PURR_w$ is a set of atoms.
This mismatch could be resolved with allowing $^D$ to be a typeshifting operation:

*three cats purr* → $APPLY[{}^{D}PURR_w, \lambda P.\exists x[{*}CAT_w(x) \wedge |x|=3 \wedge P(x)]]$

$= \exists x[{*}CAT_w(x) \wedge |x|=3 \wedge {}^{D}PURR_w(x)]]$

Some sum of three cats has the property that each of its atomic parts purrs =
There are three cats and each of them purrs.

There is no such mismatch in

*three cats meet on the roof* → $APPLY[MEET_w, \lambda P.\exists x[{*}CAT_w(x) \wedge |x|=3 \wedge P(x)]]$

$= \exists x[{*}CAT_w(x) \wedge |x|=3 \wedge MEET_w(x)]]$

Some sum of three cats has the property that it meets on the roof.

With this analysis we predict a collective reading for

*three cats carry the ball upstairs* → $APPLY[CARRY_w, \lambda P.\exists x[{*}CAT_w(x) \wedge |x|=3 \wedge P(x)]]$

$= \exists x[{*}CAT_w(x) \wedge |x|=3 \wedge CARRY(x)]]$

Since there is no mismatch, we don't predict a distributive reading.
Since there *is* a distributive reading we must make a different assumption:

**Roberts Generalization** (Craige Roberts 1991):
Distributivity is a grammatical operation that can be triggered in context.

(1) The boys carried a piano upstairs.
Out of the blue: collective

In context: In the game show, each boy needs to bring a toy piano up greased moving stairs,, while the girls hose empty a barrel of gooye stuff. What did the boys do while the girls worked on the gooye stuff? *The boys carries a piano upstairs*.

*three cats carry the ball upstairs* → $APPLY[^{D}CARRY_w, \lambda P.\exists x[{}^*CAT_w(x) \wedge |x|=3 \wedge P(x)]]$

$= \exists x[{}^*CAT_w(x) \wedge |x|=3 \wedge {}^{D}CARRY(x)]]$

**Non-associative readings**

Link 1984 (AC 1982) discussed non-associative readings as in (1)

(1) Napoleon and Blüchner and Wellington fought each other at Waterloo.

The point is that ($n \sqcup b \sqcup w$) does not express who fought who. A similar example is from Landman 1989, *Groups*:

(2) The cards below seven and the cards from seven up are separated.

The intuition is that the cards are separated into two groups, $\sigma(1\sqcup\ldots\sqcup 6)$ and $\sigma(7\sqcup\ldots\sqcup ace\}$, but you are just given one group: $\sigma(1 \sqcup \ldots\sqcup 6 \sqcup 7 \sqcup\ldots\sqcup ace)$.

Link 1984 proposes a group formation operation. I will give it the form of Landman 2000 (which improves upon Roger Schwarzschild's improvement of Link):

We assume that our set of atoms is partitioned into two groups: IND (individuals) and GROUP (groups), with IND ∩ GROUP = Ø.
*cat*, as a predicate of individuals is interpreted as:

*cat* → $CAT_w \subseteq IND$

*committee* as a predicate of groups is interpreted as:

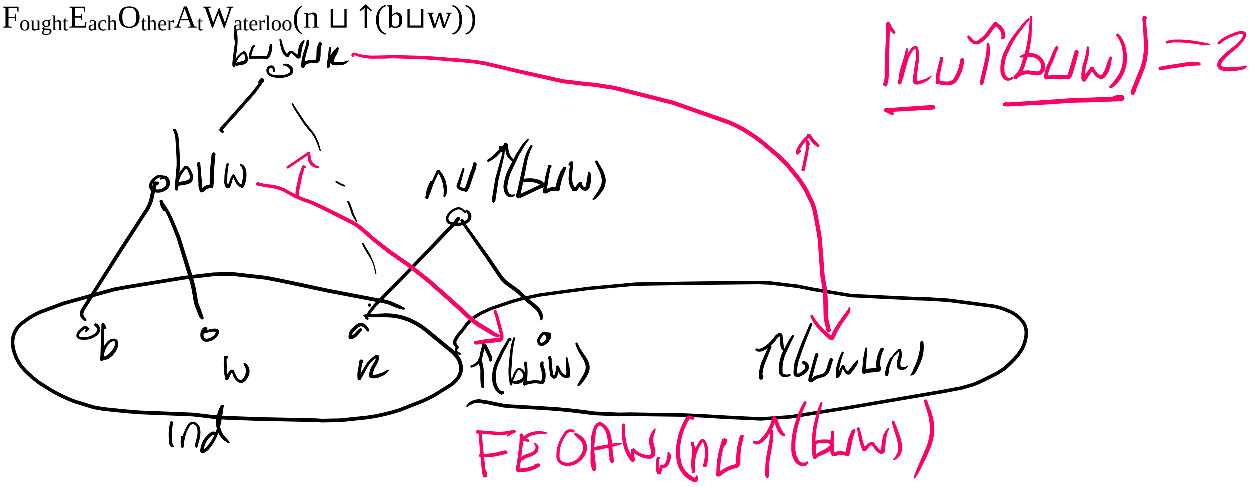
*committee* → $COMMITTEE_w \subseteq GROUP$

We specify two functions:
group formation ↑ which is a one-one function from sums of individuals to groups
and membership specification ↓ which is a function from groups to sums of individuals:
with the conditions as specified:

$\uparrow: {}^*IND \cup GROUP \rightarrow ATOM$
$\downarrow: GROUP \rightarrow {}^*IND$
where:
1. for every $a \in ATOM$: $\uparrow(a)=a$
2. for every $b \in {}^*IND - ATOM$: $\uparrow(b) \in GROUP$
3. if $x \neq y$ then $\uparrow(x) \neq \uparrow(y)$
4. for every $b \in {}^*IND$: $\downarrow\uparrow(x) = x$

We assume that ↑ and ↓ are freely available as type shifters between DP interpretations (so not at the NP level, but at the DP level). So we don't assume that *cats* naturally shifts from a set of pluralities to a set of groups. But we do assume that, say, the DP *Blüchner and Wellington* can freely shift between ($b\sqcup w$) and $\uparrow(b\sqcup w)$. This allows for

(1a) $F_{ought}E_{ach}O_{ther}A_{t}W_{aterloo}(n \sqcup \uparrow(b \sqcup w))$




The cards below seven and the cards from seven up:

$\sigma(\lambda x.CARD_w(x) \wedge \#(x)<7)$ $\sigma(\lambda x.CARD_w(x) \wedge \#(x)\geq 7)$
$\uparrow\sigma(\lambda x.CARD_w(x) \wedge \#(x)<7)$ $\uparrow\sigma(\lambda x.CARD_w(x) \wedge \#(x)\geq 7)$

$\uparrow\sigma(\lambda x.CARD_w(x) \wedge \#(x)<7) \sqcup \uparrow\sigma(\lambda x.CARD_w(x) \wedge \#(x)\geq 7)$
a sum of two groups.

(3) In this game the cards below seven and the cards from seven up are ***both*** used, but they have different roles.

Both: sum of 2

**Collectivity as semantic singularity: (Landman 1989, 2000)**

With this Link creates ambiguity of which it is not clear what it means, what the difference is between:

$MEET_w(\sigma(*CAT_w)$
$MEET_w(\uparrow(\sigma(*CAT_w))$

Landman 1989:

1. Individual level predicates:
   *purr* → $PURR_w \subseteq IND$
   Group level predicates:
   *meet on the roof* → $MEET_w \subseteq GROUP$
   Mixed predicates:
   *carry the ball upstairs* → $CARRY_w \subseteq ATOM$

2. Semantic plurality in the verbal domain:
   verbal predicates (like *purr*) are by default pluralized with *

*three cats* → $\lambda P.\exists x[*CAT_w(x) \wedge |x|=3 \wedge P(x)]$
$\lambda P.\exists x[*CAT_w(x) \wedge |x|= 3 \wedge P(\uparrow x)]$

*three cats purr* → #APPLY[$^*PURR_w, \lambda P.\exists x[*CAT_w(x) \wedge |x|= 3 \wedge P(\uparrow x)]$
Infelicitous, because $*PURR_w \subseteq *IND$ and $\uparrow x \in GROUP$

APPLY[$^*PURR_w, \lambda P.\exists x[*CAT_w(x) \wedge |x|=3 \wedge P(x)]]$ =
$\exists x[*CAT_w(x) \wedge |x|=3 \wedge *PURR_w(x)]]$
Some sum of three atomic cats is a sum of atomic purrers

*three cats meet on the roof* → #APPLY[$*MEET_w, \lambda P.\exists x[*CAT_w(x) \wedge |x|=3 \wedge P(x)]]$
Infelicitous, because $*MEET_w \subseteq *GROUP$ and $x \in *IND$

APPLY[$*MEET_w, \lambda P.\exists x[*CAT_w(x) \wedge |x|= 3 \wedge P(\uparrow x)]]$
$\exists x[*CAT_w(x) \wedge |x|=3 \wedge *MEET_w(\uparrow x)]]$ =
$\exists x[*CAT_w(x) \wedge |x|=3 \wedge MEET_w(\uparrow x)]]$
Some group consisting of three cats meets

*three cats carry the ball upstairs* → APPLY[$*CARRY_w, \lambda P.\exists x[*CAT_w(x) \wedge |x|=3 \wedge P(x)]]$
$= \exists x[*CAT_w(x) \wedge |x|=3 \wedge *CARRY_w(x)]]$
There is a sum of three atomic cats that is a sum of atomic ball carriers
APPLY[$*CARRY_w, \lambda P.\exists x[*CAT_w(x) \wedge |x|=3 \wedge P(\uparrow x)]]$
$= \exists x[*CAT_w(x) \wedge |x|=3 \wedge CARRY_w(\uparrow x)]]$
There is a group consisting of three cats that carries the ball upstairs.

**7.9 Collectivity and distributivity** [From Landman 2000, with some deletions]

Above I have made the distinction between basic, singular, predicates and plural predicates that are pluralizations of basic predicates. As I argued, these are not the only kinds of predicates that need to be distinguished. The grammar will contain, besides pluralization, operations that turn plural predicates into complex plural predicates, and the latter need not be pluralizations of basic, singular, predicates.

However, all this stays rather abstract theorizing, as long as we do not determine what counts as a basic predicate.

I think of basic predicates as by and large corresponding to those lexical items that assign thematic roles (though not all of them, some lexical items like for instance *look alike*, if that is a lexical item, I would assign internal logical structure, making it, in effect, a plural predicate).

Basic predicates are predicates that have thematic commitment.
If a basic, singular predicate applies to a certain argument, that argument fills a thematic role of that singular predicate.
This means that whatever semantic properties are associated with that thematic role,
the object that fills that role in that predication has those properties.

Thus, for example, I assume that *sing* is a basic predicate assigning the agent role to its subject.
By that, I assume that whatever semantic inferences and implicatures follow from filling the agent role of *sing*, if *John sings* is true, then those inferences and implicatures hold with respect to John.

This is rather straightforward.
But it has an interesting consequence.
I have assumed in the previous section that collective predication is an instance of singular predication.
If singular predication is thematic predication, it follows that collective predication is thematic predication.
Moreover, if collective predication is nothing but singular, thematic predication,
it follows that there is no place in the grammar for a separate theory of collective predication,
i.e. there is no separate theory of collective inferences.

This means that all inferences and implicatures associated with collective readings have to be derived from two sources:
first the general theory of thematic roles and inferences associated with those,
and second the nature of the argument filling the role,
i.e. the fact that a group, rather than an individual fills a role.

Let me give some examples.

**Example 1. Collective body formation:**

(35) a. The boys touch the ceiling.

I have discussed this example already in Lecture Three.

The example is a variant of the example of the leaves of the trees, discussed in lecture Four.
For (35a) to be true on a collective reading, there is no need for more than one boy to do the actual touching:
(35a) is true if the boys form a pyramid and the top boy touches the ceiling.

A theory of collective inferences would explain this by assuming that the predicate *touch*, as applied to a collection, distributes semantically to at least one of the members of the collection, while the involvement of the others is a matter of cancelable implicatures.

The alternative explanation that I would propose (following basically Scha 1981) is the following:
Compare (35a) with (35b):

(35) b. I touch the ceiling.

What does it mean for me to touch the ceiling?
On a direct contact interpretation, it means that part of my body is in surface contact with part of the ceiling.
This follows from the meaning of *touch*: part of the agent is in surface contact with part of the theme.

Exactly the same meaning is involved in (35a).
The only difference between (35a) and (35b) is that in (35a) it is a collection that fills the agent role and a collection has different parts from an individual:
in particular it can be individuals that are part of collections.

Thus, we do not need to assume anything special about collective predication to explain this case.

Now, the direct contact interpretation is not the only interpretation of *touch*.
We can have an instrumental interpretation, as in (36):

(36) a. The boys touch the ceiling with a stick.
b. 1 touch the ceiling with a stick.

The crucial point is, though, that the change of interpretation of *touch* from (35) to (36) is the same for the a and the b-cases.
On the most immediate interpretation, *touch* in (36) means that the agent brings a stick that it is in contact with in immediate contact with the ceiling.
We don't need to assume that *touch* has a different meaning dependent on whether the agent is a singular object or a group:
what differences there are simply follow from real world consideration concerning what it takes for an individual, or for a group, to brings a stick that it is in contact with in immediate contact with the ceiling.

**Example 2: Collective action:**

(37) a. The boys carried the piano upstairs.

In a collective action, as in the collective interpretation of (37a), the predicate does not necessarily distribute to each of the boys (the actual predicate needn't distribute at all), nor does it have to be the case that all the boys in (37a) have to be directly involved in the action,
i.e, not all boys have to do actual carrying (like the one who is walking in front with a flag).

In a normal context, (37a) will implicate some other things about the boys, like that some of them are (at least partly) under the piano some of the time, and that some of them move up the stairs. Let us again compare with the singular case (37b):

(37) b. I carried the piano upstairs.

Also in (37b), not all my parts do the carrying (my big toe doesn't).
While it tends to be the case that when I carry the piano upstairs all my parts move up the stairs, such differences can easily be attributed again to the differences in the relation between me and my parts and collections and their parts:
collections can be spatially discontinuous, my parts are not.

**Example 3: Collective responsibility:**

Lasersohn 1988 argues that often in collective readings, we do not require that the individuals are directly involved in the action, but that they do share in the responsibility:
we ascribe collective responsibility to the agent in a collective predication.
Cf. example (38a):

(38) a. The gangsters killed their rivals.

While not every gangster may have performed any actual killing, that will not help them in court: the individual gangsters are co-responsible for the killings.
Again this is not different from the singular case, cf. (38b):

(38) b. Al Capone killed his rivals.

It is a general property of agents (of verbs like *kill*) that we can assent to the truth of the sentence, even though the agent does not literally perform the action:
(38b) is true even if Capone didn't pull the trigger himself, because, if he ordered it, he bears the responsibility for the action.
The difference between (38a) and (38b) again can be described in terms of whether and how responsibility of an agent relates to responsibility of the parts:
we naturally spread the total mass of responsibility over sentient parts of an agent,
but not naturally over non-sentient parts (though Capone's bad liver may have had something to do with it).

The conclusion is: all these cases involve thematic predication where a collection fills a thematic role.
Differences with singular predicate are reduced to the differences between individuals and collections.

It now becomes interesting to compare these cases with what I have called plural predication. Look at (39) on the distributive reading:

(39) The boys sing.

The crucial difference with the previous cases is that, on the distributive interpretation of (39), it is not the entity that is the subject in the predication - the denotation of the boys - that is claimed to have the semantic properties that agents have, but the individual boys.

On the distributive reading, no thematic implication concerning the sum of the boys itself follows.
This means that the distributive predication is not an instance of thematic predication of the predicate sing to its subject.

This has an important theoretic consequence about the way we set up the grammar.
Some analyses of plurality assume that also in a distributive predication in (39), the boys fills the agent role of a basic predicate sing. This is, for instance, what Scha 1981 does for examples like (39).
Scha, and others following him, would derive the distributive reading through an optional meaning postulate on sing (i.e. on one of its meaningsX sings is equivalent to every part of X sings).
However, since there are no thematic inferences concerning what fills the agent role in the distributive reading, on such a theory, it follows that there cannot be any semantic content to the notion of agent at all.
That is, this approach is incompatible with any theory that assigns any semantic, thematic property to a thematic role, because in the distributive cases the entity that fills the role doesn't have the relevant property.

This would mean that thematic roles cannot have any content.
Not even a weak theory of thematic roles, like the one in Dowty1991, would be possible: thematic roles become meaningless labels.
Now, some might be willing to accept the impossibility of a theory of thematic roles without batting an eyelid.
My feeling is that whatever the possibility of a contentful theory of thematic roles, it is not the business of the theory of plurality to make it impossible.
This means, then, that in plural, distributive, predication in (39), the subject the boys does not fill a thematic role R of the predicate sing.

1 will assume that the subject does fill a role, and that the role that it fills is a non- thematic, plural role defmed on the real thematic role R.
I assume then that thematic predication is predication of a thematic basic predicate to an argument, predication where the argument fills a thematic role of a basic predicate.
We think of basic predicates here as predicative lexical items and of thematic roles as lexical generalizations concerning their arguments.

What I have been arguing is something very simple, which tends to get ignored.
As said, theories of thematic roles are theories about generalizations over lexical properties assigned by predicates to their arguments.
These theories hardly ever take plurality into account.
Thus, they will tell you, for instance, that a particular kind of verb assigns the agent role to its subject.
And this is usually taken to be a lexical-syntactic principle, rather than a semantic principle.

But if that is so, then the boys in (39) will be the agent of *sing*, independently of whether *sing* is interpreted distributively or collectively.
But that means that the denotation of the boys, which is the sum of the boys, should be an agent, independently of whether *sing* is interpreted distributively or collectively. But on the distributive interpretation, not a single lexical property that you might want to single out as a candidate for being part of agenthood is predicated of the denotation of the boys, the sum of the boys itself, in virtue of the boys being the agent of sing!

This means that we are left with two choices:
give up lexical semantics, or give up the idea that the boys is the agent of sing in the distributive reading.
I am advocating the second strategy:
on the distributive interpretation of *sing*, the denotation of the subject the boys does not fill the agent role of *sing* (which is the thematic role of agent with lexical content determined by sing),
but it fills a plural agent role of the semantically plural predicate, which is the pluralization of the basic predicate *sing*.
The pluralization of *sing* is not a basic lexical predicate, but a derived predicate, defined in terms of the basic lexical predicate *sing*.
Similarly, the plural agent role that the interpretation of *the boys* fills is not a lexical thematic role, but a derived role, defined in terms ofthe basic lexical thematic role.

Let us now come back to collectivity.
In theories of plurality, including my own, a lot of unclarity exists about what counts as a collective reading and how to distinguish collective readings from non-collective readings.

The framework that I am developing here suggests a criterion for determining when we are dealing with a collective reading:

**The collectivity criterion:**
The predication of a predicate to a plural argument is collective iff the predication is predication of a thematic basic predicate to that plural argument,
i.e. is a predication where that plural argument fills a thematic role of the predicate.

One side of this criterion is the proposal that I made before:
there is not a special theory of collectivity implications;
collectivity implications are instances of thematic implications:

The presence of collectivity implications indicates that the predication is thematic predication.

I think that, though not much discussed explicitly in this form, this part of the collectivity criterion is widely accepted in the literature on plurality.
What turns it into a criterion is the other direction:

Lack of collectivity implications indicates non-thematic predication.

This part tells us that we cannot use the notion of collectivity as a plural waste-paper basket. It tells us that if a certain predication arguably lacks collectivity implications,
we cannot subsume it under collective predication,
and this means that we cannot leave it unanalyzed:
we have to assume that it is a complex plural predicate, derived from other predicates through the plurality operations that are available in the grammar (of which simple pluralization of a basic predicate is an instance).

The interesting thing is that when one reads the literature
(for instance the literature arguing in favor of a distributive operation, e.g. Link 1984, Roberts 1987, Hoeksema 1987, Landman 1989a, and many others),
one gets the impression that also this part is widely accepted (though implicitly) in the literature.
If so, then it is more accepted as a virtuous ideal than as a matter of praxis, because I don't know a single theory that actually manages to live up to this part.
And the reason is that, while it has the great advantage of clarifying the notion of collectivity, it makes life difficult.

For instance, many cases that I regarded in Landman 1989a as instances of collective predication that maybe the lexicon, but not the grammar, needs to analyze further, can no longer be regarded as such.
Applying the collectivity criterion to predicates like *look alike*, *separate* or *sleep in different dorms* tells us that these predicates cannot be regarded as collective predicates,
and shows (in line with Roberts 1991, Schwarzschild 1991, 1996, Carlson 1987, Moltmann 1992) that we need to regard these as complex plural predicates.
Let us go through some of these cases in some detail.
I argued above that, on the dependent reading, (40) is not distributive:

(40) The boys kissed different girls.

However, on the reading that we are interested in, (40) is (roughly) equivalent to (41):

(41) Each boy kissed a different girl.

When we think about agentivity, we see that, on this interpretation, the sum of the boys in (40) is as little the agent of kiss different girls as it is in (41):
(40) and (41) express, on the relevant reading, that kissing is going on between individual boys as agents and individual girls as themes.
(40) and (41) do not ascribe any agentive property to the sum of the boys, nor any theme property to the sum of the girls.
What the dependent plurality tells us is that these individual kissings take place in a certain distribution, i.e. the plurality involves a comparison between individual kissings.
Since there is no evidence that in this case the boys is the lexical agent of kiss, by the collectivity criterion, we should not assume that in this case the sum of the boys fills a thematic role of the basic predicate KISS, but rather that it fills a non-thematic role of a complex plural predicate.

The same argument can be made for predicates like look alike in (42):

(42) The boys look alike.

This predicate does not ascribe any the1natic property directly to the sum of the boys, rather it compares certain features of the individual boys, and expresses that there is a certain distribution of certain of those features (like: each boy's cmcial facial features fall within a certain class). Consequently, the collectivity criterion tells us that the relevant reading of (42) is not the collective reading -where the boys fills a thematic role, but is a plural reading - where the boys fills a complex non-thematic plural role.

While I am aware that the application of the collectivity criterion is difficult in various subtle cases, and while I am not sure that in a fmal account (if there is such a thing) the criterion can hold unmodified, I think (and hope to show) that the criterion has very interesting empirical and theoretical consequences, and provides a healthy methodology for determining what should be part ofthe grammatical theory of plurality and what can be left to the lexicon.

**7.10 Partial distributivity and distributivity**

Consider the following examples from the literature:

(43) The marines invaded Grenada. [Carlson 1977]
(44) The leaves of tree A touch the leaves of tree B. [Similar to examples in Scha 1981]
(45) The journalists asked the president five questions. [Dowty 1986]

These are all cases where for the sentence to be true, it is not necessary that all individual parts of the plural entities involved have the relevant property, invading, touching, asking.

But it does seem to be necessary that *some* individual parts have the relevant property.
One could call this phenomenon *partial distributivity*:
the relevant property distributes to some, but not necessarily all parts of a plural argument.

Since total distributivity is the borderline case of partial distributivity, o
ne could use this as an argument to eliminate distributivity from the grammar.
The argument would go as follows:

the lexical meaning of the predicates in (43)-(45) tells us that some parts have the relevant property. Total distributivity is compatible with that.
The so-called distributive reading is not a separate reading, but an instance of the basic reading; i.e. it represents one type of situation in which the basic reading of (43)-(45) is true.

Given this, we would not need a separate distributive operator or a pluralization operator.

As we have seen, this position is what Scha 1981 assumes for definites.
Hoeksema 1987 could be interpreted as defending a similar view.
Dowty 1986 was the fırst to point at example (45) in the context of his discussion of distributive sub-entailments, though it is not clear to me that Dowty 1986 actually proposes to reduce distributivity to partial distributivity.

As can be seen from the caution in the above lines, it isn't clear to me that this proposal has actually been explicitly defended in the literature (Irene Heim once proposed it to me, but p.c.)

The problem with this proposal is that in all cases (43)-(45), there are collective, thematic implications.
An other way to say this is that all these cases are non-inductive.

Take (43).
Suppose that two members of the Marine Corps in a totally unauthorized action land on Grenada. Would we say that this is sufficient to make (43) true?
I don't think so. Now, maybe if we increase the numbers of Marines landing we reach at a certain moment a point where we're inclined to count (43) as true.
But crucially, this is *not* because of the numbers, but because at a certain moment, this becomes an action that reflects on the whole Marine Corps, a *collective responsibility*.
And it is this collective responsibility, rather than the number of individual Marines involved, that makes (43) true.

\Similarly, in (44), it is because we easily conceive of the leaves on a tree as a coherent body (the foliage) (and there is surface contact) that we regard (44) as true.

(45) is a funny sentence:

(45) The green leaves in Holland touch the yellow leaves in Holland.

If (44) were inductive, then the touching of two individual leaves would be sufficient to make (44) true. But then there is no reason why (46) shouldn't be a perfectly fine and true sentence.

But (46), out of the blue, is weird, and the reason is collectivity:
out of the blue it is difficult to turn the green and the yellow leaves in Holland into coherent bodies (coherent enough to make 'touching' a sensible relation between them).

(45) may seem more inductive: if five individual journalists happen to ask a question, isn't that sufficient to call (45) true?
While it is harder to detect, 1 think that also (45) is in fact not inductive and involves collective responsibility.
A press conference is a kind of allegorical play with fixed roles fllled by certain individuals or groups (The President, The Press);
it is the business of journalists at a press conference to ask questions, and it is part of the play that this is done in a certain way (distributing question asking over joumalists).
Yet, (45) makes an evaluative statement about the functioning of the whole Press corps:
they got in five questions, which means, depending on the kind ofpress conference, that THEY (the press) did or didn't do their job well.
Another reason to assume that (45) involves a collective reading is the following.
Look at (47):

(47) The press asked the president five questions.
(47), 1 think, does not differ from (45) at all in how inductive or non-inductive it is.
But (47) involves a singular collective expressions.
While Schwarzschild 1991 argues convincingly (against Landman 1989a) in favor of distinguishing such singular collective expressions from plurals in collective readings,

there doesn't seem to be a difference here, and since also for Schwarzschild, singular collective expressions are as prototypically collective as you get, this provides another reason to assume that partial distributivity is a collectivity effect.

In all these cases, then, there are collectivity implications.
By applying the collectivity criterion, this means that in all the cases (43)-(45) we are in fact dealing with collective readings.

Now compare the previous cases with the distributive interpretation of (8):

(8) The boys carried the piano upstairs.

Think of the following context: in a game show the girls each have to swim fifteen meters, while each of the boys carries a toy-piano upstairs (which then, each time, is brought down again for the next boy. To make it difficult, we assume that the stairs are greased.).
In that context, the distributive interpretation of (8) is fine after the following question:
What were the boys doing while the girls were swimming?

The important thing here is that, on this interpretation, (8) is purely inductive:
if boy 1 carried the piano upstairs,...., boy n carried the piano upstairs, then we can truthfully say (8) on the distributive interpretation.

This means that (8) does not involve any thematic implications concerning the plural argument of the predication, the boys, itself.
Applying the collectivity criterion, it follows that (8) is not a collective reading, and does not involve a basic, thematic predicate.

It follows then from the collectivity criterion, that the attempt to eliminate distributivity from the grammar by trying to reduce it to lexical partial distributivity fails: partial distributivity is collectivity, and hence thematic predication, while true distributivity is non-thematic predication.

Now let us look at (48):

(48) Forty journalists asked the president only seven questions.

(48) looks like a cumulative reading, except that, because of the distribution of the numbers (assuming that questions don't get asked more than once), if it is a cumulative reading, it can only be a partial cumulative reading.
But (48) isn't different from (45), so by the same argument as before, we should conclude that (48) is a collective reading, and that the partial cumulativity effect is really a thematic, collectivity implication.

More evidence for this is example (49):

(49) Fifteen women gave birth to only seven children.

Out of the blue, (49) is weird.
The natural reaction to (49), out of the blue, is something like: how did they manage to do that? There is a natural explanation for this: *give birth to* - as a relation between women and children - is a relation that strongly resists its first argument to be interpreted collectively: *give*

*birth to* is a hyper-individual relation between a woman and a child: we do not think of women as giving birth to children in groups.
Thus, (49) does not, out of the blue, have a collective reading, and that's why it is weird.

Compare (49) with (50):

(50) Seven hundred chickens laid fifty eggs.

Unlike (49), (50) is not weird out of the blue.
In the context of what is called in Dutch the Bio Industry, (50) can easily be interpreted as a comment on the malfunctioning of a particular chicken battery.
The reason is that the role of chickens in a battery is similar to that of journalists at a press conference: who cares that also a chicken has a hyper-individual relation to her egg:
for us, chickens are means of producing eggs: it is the business of chickens in a battery to produce a certain quota of eggs. For that reason, it is easy for us to ascribe collective responsibility to the chickens in a certain battery for the malfunctioning of the battery.
And for that reason, a collective reading of (50) is easily available.

This is much more difficult in (49). b
I'm not claiming that (49) doesn't have a collective reading:
for instance in the not so natural context of hospital statistics, (49) can get a collective reading as well, and it can be seen as a comment on the malfunctioning of the matemity ward.
Nevertheless, out of the blue this reading is not available for (49).

But now look at (51):

(51) Seven women gave birth to fifteen children.

There is a sharp difference, out of the blue, between (49) and (51).
Unlike (49), (51) is fine, and the reading that is in fact most prominent is the cumulative reading: seven women gave birth to children and fifteen children were born to them.
These facts form a serious problem for Roberts 1987's proposal to reduce cumulative readings to collective readings.
If cumulative readings are collective readings, then the cumulative reading of (51) should be a collective reading, like (49).
But we have seen that (49), out of the blue, doesn't have a collective reading:
(49) is weird out of the blue.
But if (49) doesn't have a collective reading out ofthe blue, neither should (51):
i.e. (51) should be just as weird as (49).
But (51) isn't weird out of the blue, it is fine.
Hence we have a strong argument here that cumulative readings are in fact not collective readings.

In sum: (49) is weird because it does not naturally have a collective reading and it cannot have a cumulative reading (because of the numbers).
(51) is weird on a collective reading, just like (49), but it is fine on a cumulative reading.

A second argument that cumulative readings should not be reduced to collective readings involves the collectivity criterion directly.
Cumulative readings are as inductive as distributive readings (as argued extensively in Krifka 1989a, 1989b, and other work by Manfred Krifka).

If Sarah gave birth to Chaim and to Rakefet, and Hannah gave birth to Avital, we can truthfully say (52) on a cumulative reading:

(52) Sarah and Hannah gave birth to Chaim, Rakefet and Avital.

The collectivity criterion then tells us that cumulative readings are non-thematic, and hence non-collective.
This means that Roberts' attempt to eliminate cumulativity from the grammar by reducing it to collectivity fails, and we have strong arguments that the grammar needs to deal with cumulative readings after all.
In fact, cumulative readings are not like collective readings at all, rather they are very closely related to distributive readings.
Hence, if we want to reduce cumulative readings to something else, it shouldn't be collectivity, but plurality.

At this point I will temporarily stop the discussion of plurality.
What follows is a section on mass nouns (which I am skipping for now).
I will continue the discussion of plurals and cumulativity in the next chapter in the context of event theory.

## 7.11. Boolean semantics for mass nouns

Link 1983 proposes a semantics in which mass and count nouns are interpreted in distinct but linked domains. Here I will give a version close to that in Landman 1991:

▹ A *Boolean interpretation domain* is a structure $\boldsymbol{B} = \langle \mathbf{B},\mathbf{M},\mathbf{C}, \downarrow,\uparrow \rangle$, where:

1. **B** is a complete Boolean algebra such that $B = {}^*(M \cup C)$.
2. **M**, *the mass domain*, is a complete Boolean algebra.
3. **C**, *the count domain*, is a complete atomic Boolean algebra.
4. $0_{\mathbf{B}} = 0_{\mathbf{M}} = 0_{\mathbf{C}}$.
5. $\downarrow$, *the grinding function*, is a function from C into M such that:
   for every $x \in C$: $\downarrow x = \sqcup_{\mathbf{M}}\{\downarrow a: a \in ATOM_x\}$.

   Grinding maps a count element onto the sum of its mass parts.
6. $\uparrow$, *the portioning function*, is a one-one function from M into $ATOM_C \cup \{0\}$ such that:
   $\uparrow(0) = 0$ and for every $x \in M$: $\downarrow(\uparrow(x)) = x$.

   Portioning treats a mass object as an atomic count object.

▹ A *Boolean interpretation structure* is a structure $\boldsymbol{I} = \langle \boldsymbol{B},W \rangle$, where $\boldsymbol{B}$ is a Boolean interpretation domain and W is a set of indices.

Intensions are functions $P: W \rightarrow \mathbf{pow}(B)$

Two useful notions are *cumulativity* and *homogeneity*:

Let $X \subseteq B$
▹ X is *cumulative* iff if $X \neq \emptyset$ then $X = {}^*X$.
▹ X is *homogenous* iff if $X \neq \emptyset$ then $X = (\sqcup X]$.

A set is cumulative if it is identical to its own closure under sum. A set is homogenous if it is identical to its own part set.

**Lemma:** If X is homogenous, then X is cumulative.
If $X \subseteq ATOM_{\mathbf{B}}$, then $^*X$ is homogenous

Let $P: W \rightarrow \mathbf{pow}(B)$ be an intension.
▹ P is *cumulative* iff for every $w \in W$: $P_w$ is cumulative.
▹ P is *homogenous* iff for every $w \in W$: $P_w$ is homogenous.

Lønning 1987 gives an analysis in which *lexical mass nouns* are interpreted as *homogenous* intensions.
Krifka 1986, 1989 proposes that lexical mass nouns be interpreted as *cumulative intensions*.
In both cases the semantics explores analogies between the denotations of lexical mass nouns and *lexical plural nouns*. For Lønning both are homogenous, for Krifka both are cumulative.

Example:

*Lexical mass noun*: *wine* → $\text{WINE}_w$,
where $\text{WINE}_w$ is a cumulative subset of M .

$\text{WINE}_M$ is cumulative means that $\text{WINE}_w = {}^*\text{WINE}_w$.
Hence it follows that $\sqcup\text{WINE}_w \in \text{WINE}_w$.
And this means that σ is defined for $\text{WINE}_w$:

*the wine* → $\sigma(\text{WINE}_w) = \sqcup\text{WINE}_w$ The sum of the wine

Cumulativity is shown for plurals in (15) and for mass nouns in (16):

Let *cat* → $\text{CAT}_w$ *pet of mine* → $\text{MY-PET}_w$
*pet of yours* → $\text{YOUR-PET}_w$
*Marc* → $\text{MARC}_w$ *liquid in my glass* → $\text{MY-LIQUID}_w$
*liquid in my glass* → $\text{YOUR-LIQUID}_w$

where $\text{CAT}_w$, $\text{MY-PET}_w$, $\text{YOUR-PET}_w \subseteq \text{ATOM}_C$
and $\text{MARC}_w$, $\text{MY-LIQUID}_w$, $\text{YOUR-LIQUID}_w$ are cumulative subsets of M.

(15b) and (16b) are provably valid inferences:

(15) a. If my pets are cats and your pets are cats then my pets and your pets are cats.
b. $^*\text{CAT}_w(\sigma(^*\text{MY-PET}_w)) \wedge {}^*\text{CAT}_w(\sigma(^*\text{YOUR-PET}_w)) \Rightarrow$
$^*\text{CAT}_w(\ \sigma(^*\text{MY-PET}_w) \sqcup \sigma(^*\text{YOUR-PET}_w)\ )$

(16) a. If the liquid in my glass is Marc de Bourgogne and the liquid in your glass is Marc de Bourgogne, then the liquid in my glass and the liquid in your glass is Marc de Bourgogne.
b. $\text{MARC}_w(\sigma(\text{MY-LIQUID}_w)) \wedge \text{MARC}_w(\sigma(\text{YOUR-LIQUID}_w)) \Rightarrow$
$\text{MARC}_w(\ \sigma(\text{MY-LIQUID}_w) \sqcup \sigma(\text{YOUR-LIQUID}_w)\ )$

Two special cases of Boolean interpretation domains are discussed in the literature, the Boosk domain and the Classical domain:

*The Boosk domain* [Link 1983]
▻ A *boosk* is a Boolean interpretation domain where ↑ is the identity function.

For Link 1983 all mass objects are themselves atomic count objects (
so the mass partial order $\sqsubseteq_M$ is an order between objects that are themselves atoms with respect to the count partial order $\sqsubseteq_C$).

*The classical domain* [Landman 1991]
▻ A *classical domain* is a Boolean interpretation domain where M is atomless and $C^+ \cap M^+ = \emptyset$.

If the mass domain is atomless and the denotation $\text{MUD}_w$ of mass noun *mud* is homogenous, then $\text{MUD}_w$ is *divisible*:

▻ X is *divisible* iff $\forall x \in X^+ \exists x_1 \in X^+ \exists x_2 \in X^+ [x = x_1 \sqcup x_2 \wedge x_1 \sqcap x_2 = 0]$.

If $MUD_w$ is divisible, then every object that counts as mud can be partitioned into two objects that also count as mud. Count noun denotations cannot satisfy divisibility, since divisibility stops at atoms.
The idea that mass noun denotations differ from count noun denotation in that mass noun denotations satisfy some sort of divisibility and hence are not built from minimal elements or don't have minimal elements at all, was common in the earlier ('classical') semantic literature (e.g. ter Meulen 1980, Bunt 1985, Link 1983, Landman 1991).
A representative example is given by the following (almost) quote:

> "What are the minimal parts of water? Chemistry tells us that they are the water molecules. But water molecules can be counted, while water cannot be counted. This shows that natural language semantics does not incorporate the insights of chemistry in its models: in our semantic domains, the water molecules are not the minimal parts of water. In fact, the real semantic question is: is there any evidence, semantic evidence, to assume that mass entities like water are built from minimal parts at all, either from minimal parts that are water, or from minimal parts that aren't water? If there is no such semantic evidence, it is theoretically better to assume that the semantic system does not impose a requirement of minimal parts.
> Since there is no semantic evidence for minimal parts, we should assume non-atomic structures for the mass domain. That has the added bonus that we can nicely explain why we cannot count mass entities, because counting is counting of atoms." (paraphrase of Landman 1991, pp 312-313)

We defined earlier:

▻ *Count sets*: Z is *count* iff if $Z^+ \neq Ø$ then **(**Z**]** is a complete *atomic* Boolean algebra.
▻ *Count intensions*: P is *count* iff for every $w \in W$: $P_w$ is count.
▻ *Count NPs*: NP α is *count* iff for every Boolean interpretation structure and interpretation function, α is interpreted as a *count* intension.

For lexical count nouns, like *cat*, the notion *count* takes the form of a stipulation:

*Lexical constraint:* Interpret lexical count nouns as count intensions.

Given this, for complex noun phrases like *cats* or *three blind mice* you don't have to stipulate that they are count: you can actually *prove* that they are.

For mass nouns there are different options.

▻ *Z* is *non-atomic* iff if $Z^+ \neq Ø$ then **(**Z**]** is a complete *non-atomic* Boolean algebra.
▻ *Z* is *atomless* iff if $Z^+ \neq Ø$ then **(**Z**]** is a complete *atomless* Boolean algebra.

And we can identify mass with the first or the second notion:

▻ Z is *non-atomic-mass* iff Z is non-atomic.
▻ Z is *atomless-mass* iff Z is atomless.

And this generalizes to intensions and noun phrases:

▻ *Mass intensions*: P is *mass* iff for every $w \in W$: $P_w$ is mass.
▻ *Mass NPs*: NP α is *mass* iff for every Boolean interpretation structure and interpretation function, α is interpreted as a *mass* intension.

*Lexical constraint:* Interpret lexical mass nouns as mass intensions.

**Lemma:** In a classical Boolean interpretation structure:

Intensions in $(W \rightarrow \mathbf{pow}(C))$ are count.

Intensions in $(W \rightarrow \mathbf{pow}(M))$ are atomless mass.

Intensions in $(W \rightarrow \mathbf{pow}(B))$ that are neither in $(W \rightarrow \mathbf{pow}(C))$ nor in $(W \rightarrow \mathbf{pow}(M))$ are non-atomic-mass.

**Proof:** Let ***B*** be a classical interpretation domain.
-If $Z \subseteq C$ and $Z^+ \neq \emptyset$ then $(Z]$ is a complete atomic Boolean algebra.
-If $Z \subseteq M$ and $Z^+ \neq \emptyset$ then $(Z]$ is a complete atomless Boolean algebra.
-If $Z^+ \cap M \neq \emptyset$ and $Z^+ \cap C \neq \emptyset$ and $Z^+ \neq \emptyset$ then $(Z]$ is a complete non-atomic Boolean algebra which is neither atomic nor atomless.

We look at counting, count comparison and distribution in classical interpretation domains. The classical theory most directly fits the idea that these three phenomena be regarded as *diagnostics* for count nouns.

**1. *Counting:***
Count nouns can be modified felicitously by numerical phrases, mass nouns cannot.

(17) a. ✓one fish /✓two fish
b. # one mud / # two mud

| | | | | |
|---|---|---|---|---|
| *two* | + | *fish*$_{[plur]}$ | $\rightarrow$ | ✓*two fish* |
| $\lambda x.\|ATOM_x\|=2$ | $\cap$ | $*FISH_w$ | = | $\lambda x.*FISH_w(x) \wedge \|ATOM_x\|=2$ |

| | | | | |
|---|---|---|---|---|
| *two* | + | *mud*$_{[mass]}$ | $\rightarrow$ | *#two mud* |
| $\lambda x.\|ATOM_x\|=2$ | $\cap$ | $MUD_w$ | = | $\emptyset$ |

In the classical theory, $ATOM_{MUD_w} = \emptyset$,
hence the intersection with the interpretation of *two* is always empty,
i.e. the intension of *two mud* would be $\lambda w.\emptyset$, the constant function on $\emptyset$.

In the classical theory, where the intension of *mud* is an atomless-mass intension, we can formulate this as infelicity, by using the extensional presuppositional definition of *two*:

$$two \rightarrow \lambda P.\begin{cases} \lambda x.P(x) \wedge |x|=2 & \text{if } P \text{ is a } count \text{ set} \\ \bot & \text{otherwise} \end{cases}$$

If $\lambda w.MUD_w$ is an atomless-mass intension, then for every $w \in W$: $MUD_w$ is atomless,
hence for no $w \in W$: $MUD_w$ is a count set.
Hence, *two mud* is undefined.

***2. Distributivity*:** Count DPs combine with distributive predicates, mass DPs do not.

(19) a. ✓The cats have each eaten a can of tuna.
b. #The mud has each sunk to the bottom.

$$\triangleright^D \to \lambda Q\lambda x.\begin{cases} ATOM_x \subseteq Q & \text{if } x \in C \\ \bot & \text{otherwise} \end{cases}$$

*have eaten a can of tuna* → $EAT_w$
*have each eaten a can of tuna* → $^{D}EAT_w$

where:

$$^{D}EAT_w \to \lambda x.\begin{cases} ATOM_x \subseteq EAT_w & \text{if } x \in C \\ \bot & \text{otherwise} \end{cases}$$

(19a) → $ATOM_{\sigma(^*CAT_w)} \subseteq EAT_w$ =
$\forall x[CAT_w(x) \to EAT_w(x)]$

since $\sigma(^*CAT_w) \in C$.
*have sunk to the bottom* → $SANK_w$
*have each sunk to the bottom* → $^{D}SANK_w$

where:

$$^{D}SANK_w \to \lambda x.\begin{cases} ATOM_x \subseteq SANK_w & \text{if } x \in C \\ \bot & \text{otherwise} \end{cases}$$

(19b) is infelicitous, since $\sigma(MUD_w) \notin C$.

The Classical theory presents a picture of the mass-count distinction that is very crisp and clear.
Too crisp and too clear: the above diagnostics are not in fact secure as diagnostics for the mass/count distinction.
Iceberg Semantics, Landman 2020 (and my seminar)